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A new limit of the $AdS_5 \times S^5$ sigma model

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Abstract: Using the pure spinor formalism, a quantizable sigma model has been constructed for the superstring in an $AdS_5 \times S^5$ background with manifest $PSU(2,2|4)$ invariance. The $PSU(2,2|4)$ metric g_{AB} has both vector components g_{ab} and spinor components $g_{\alpha\beta}$, and in the limit where the spinor components $g_{\alpha\beta}$ are taken to infinity, the $AdS_5 \times S^5$ sigma model reduces to the worldsheet action in a flat background.

In this paper, we instead consider the limit where the vector components g_{ab} are taken to infinity. In this limit, the $AdS_5 \times S^5$ sigma model simplifies to a topological A-model constructed from fermionic N=2 superfields whose bosonic components transform like twistor variables. Just as d=3 Chern-Simons theory can be described by the open string sector of a topological A-model, the open string sector of this topological A-model describes d=4 N=4 super-Yang-Mills. These results might be useful for constructing a worldsheet proof of the Maldacena conjecture analogous to the Gopakumar-Vafa-Ooguri worldsheet proof of Chern-Simons/conifold duality.

Keywords: AdS-CFT Correspondence, Superstrings and Heterotic Strings, Topological Strings.

Contents

1. Introduction

Maldacena's conjecture [1] relating d=4 N=4 super-Yang-Mills and the superstring on $AdS_5 \times S^5$ has been verified in various limiting cases. However, in the limit where d=4 N=4 super-Yang-Mills is weakly coupled, it has been difficult to verify the conjecture because the $AdS_5 \times S^5$ background is highly curved. Although there exists a quantizable sigma model description of the superstring in an $AdS_5 \times S^5$ background using the pure spinor formalism [2], the sigma model naively becomes strongly coupled when the $AdS_5 \times S^5$ radius goes to zero.

In an $AdS_5 \times S^5$ background, the sigma model action using the pure spinor formalism has the form [2 − 5]

$$S = \frac{1}{\Lambda}\int d^2z \left[\frac{1}{2}\eta_{ab}J^a\overline{J}^b + \eta_{\alpha\widehat{\beta}}\left(\frac{3}{4}J^{\widehat{\beta}}\overline{J}^{\alpha} - \frac{1}{4}\overline{J}^{\widehat{\beta}}J^{\alpha}\right) + \text{ghost contribution}\right] \tag{1.1}$$

where J^a for $a = 0$ to 9 and $(J^\alpha, J^{\widehat\beta})$ for $\alpha, \widehat\beta = 1$ to 16 are bosonic and fermionic $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ currents constructed from the worldsheet Green-Schwarz variables $(x, \theta, \widehat\theta)$ as in the Metsaev-Tseytlin construction [6], η_{ab} is the d=10 Minkowski metric and $\eta_{\alpha\widehat\beta} = (\gamma^{01234})_{\alpha\widehat\beta}$. BRST invariance together with $PSU(2,2|4)$ invariance uniquely fixes the relative coefficients in the action, so the $AdS_5 \times S^5$ radius r only appears in the action through the sigma model coupling constant $\Lambda = \alpha'/r^2$ where α' is the inverse string tension. So the sigma model seems to be strongly coupled when the $AdS_5 \times S^5$ radius is small. However, this conclusion may be too naive since it assumes that the $PSU(2,2|4)$ algebra remains undeformed when the $AdS_5 \times S^5$ radius is taken to zero.

One limit of the sigma model which is well-understood is the d=10 flat space limit where the $AdS_5 \times S^5$ radius goes to infinity. Naively, one would go to the flat space limit by simply taking $\Lambda \to 0$, however, this limit would preserve $PSU(2,2|4)$ invariance instead of the desired d=10 super-Poincaré invariance. The correct way to go to the flat space limit is to rescale the spinor component of the $PSU(2,2|4)$ metric $g_{\alpha\widehat\beta} = \eta_{\alpha\widehat\beta}$ to

$$g_{\alpha\widehat\beta} = r\eta_{\alpha\widehat\beta} \tag{1.2}$$

in the sigma model action of (1.1), together with an appropriate rescaling of the $PSU(2,2|4)$ structure constaints. In the limit where r goes to infinity, the $PSU(2,2|4)$ algebra is deformed into the d=10 super-Poincaré algebra and the second-order kinetic term for the fermions in (1.1) blows up. Nevertheless, this limit can be taken smoothly by writing the second-order kinetic term $r\eta_{\alpha\widehat\beta} J^{\widehat\beta}\overline{J}^\alpha$ as the first-order kinetic term $\overline{J}^\alpha d_\alpha + J^{\widehat\beta}\widehat{d}_{\widehat\beta} + r^{-1}\eta^{\alpha\widehat\beta} d_\alpha \widehat{d}_{\widehat\beta}$ where d_α and $\widehat{d}_{\widehat\beta}$ are auxiliary fermionic variables. In the limit where $r \to \infty$, one obtains a first-order action for the worldsheet fermions $(\theta^\alpha, d_\alpha)$ and $(\widehat\theta^{\widehat\beta}, \widehat{d}_{\widehat\beta})$, which is the flat space version of the worldsheet action using the pure spinor formalism.

Since the structure constants of the algebra are related to the superspace torsions $T_{AB}{}^C$, this limiting procedure can be understood as a rescaling of the $AdS_5 \times S^5$ superspace torsions into the flat superpace torsions. In an $AdS_5 \times S^5$ background, $T_{\alpha a}{}^{\widehat\beta}$ and $T_{\alpha\beta}{}^a$ are non-vanishing torsions which are related by $T_{\alpha a}{}^{\widehat\beta}\eta_{\beta\widehat\beta} = T_{\alpha\beta}{}^b\eta_{ab}$. On the other hand, in a flat background, $T_{\alpha\beta}{}^a$ is non-vanishing and $T_{\alpha a}{}^{\widehat\beta} = 0$. The rescaling of the structure constants and $g_{\alpha\widehat\beta}$ as in (1.2) rescales the torsions such that

$$\frac{T_{\alpha\beta}{}^b\eta_{ab}}{T_{\alpha a}{}^{\widehat\beta}\eta_{\beta\widehat\beta}} = r. \tag{1.3}$$

So when $r \to \infty$, $T_{\alpha a}{}^{\widehat\beta} \to 0$ which corresponds to flat space.

In this paper, we will consider a different limit of the $AdS_5 \times S^5$ sigma model in which, instead of the spinor component of the $PSU(2,2|4)$ metric $g_{\alpha\widehat\beta}$ being rescaled, the vector component g_{ab} will be rescaled as

$$g_{ab} = r^{-1}\eta_{ab}. \tag{1.4}$$

Furthermore, the $PSU(2,2|4)$ structure constants will be rescaled such that in the limit where $r \to 0$, the $PSU(2,2|4)$ superalgebra is deformed into an $\mathrm{SU}(2,2) \times SU(4)$ bosonic algebra with 32 abelian fermionic symmetries. This corresponds to rescaling the torsions such that (1.3) remains satisfied when $r \to 0$, which implies that the resulting background has non-vanishing $T_{\alpha a}{}^{\widehat{\beta}}$ but has $T_{\alpha\beta}{}^{a} = 0$. Since the usual construction of supergravity backgrounds assumes that $T_{\alpha\beta}{}^{a} = \gamma^{a}_{\alpha\beta}$ [7], this $r \to 0$ limit does not correspond to a standard supergravity background.

Nevertheless, the resulting sigma model action when $T_{\alpha\beta}{}^{a} \to 0$ is very simple and can be expressed as a linear N=2 sigma model constructed from 16 chiral and antichiral N=2 superfields denoted by Θ^{rj} and $\overline{\Theta}_{jr}$, where $r = 1$ to 4 are SU(2, 2) indices and $j = 1$ to 4 are SU(4) indices. Unlike the bosonic superfields in standard N=2 sigma models, Θ^{rj} and $\overline{\Theta}_{jr}$ are fermionic superfields. It is interesting that in the open-closed matrix model duality of [8], the matter variables are also described by fermions with a second-order kinetic action. The lowest components of Θ^{rj} and $\overline{\Theta}_{jr}$ are linear combinations of the θ and $\widehat{\theta}$ variables, and the bosonic components of Θ^{rj} and $\overline{\Theta}_{jr}$ are twistor-like combinations of the ten x's and 22 pure spinor ghosts. Just as the fermionic variables had a first-order kinetic action in the flat space sigma model obtained by rescaling (1.2), the bosonic variables now have a first-order kinetic action in the N=2 sigma model obtained by rescaling (1.4).

Moreover, this N=2 sigma model is twisted as an A-model where the pure spinor BRST operator from the original $AdS_5 \times S^5$ sigma model acts in the usual topological manner as the scalar worldsheet supersymmetry generator. So the N=2 sigma model is a topological A-model with the worldsheet action

$$S = \int d^2z d^4\kappa \, \overline{\Theta}_{jr} \Theta^{rj} \tag{1.5}$$

where $(\kappa_+, \overline{\kappa}_+, \kappa_-, \overline{\kappa}_-)$ are the Grassmann parameters of the N=(2,2) superspace. This model is invariant under the bosonic isometries SU(2, 2) $\times$ SU(4) $\times$ U(1) which act on the superfields as

$$\delta\Theta^{rj} = i\Lambda^r_s \Theta^{sj} + i\Theta^{rk}\Omega^j_k + i\Sigma\Theta^{rj}, \quad \delta\overline{\Theta}_{jr} = -i\overline{\Theta}^{js}\Lambda^s_r - i\Omega^k_j\overline{\Theta}_{kr} - i\Sigma\overline{\Theta}_{jr}, \tag{1.6}$$

where $(\Lambda^r_s, \Omega^k_j, \Sigma)$ are constant parameters satisfying $\Lambda^r_r = \Omega^j_j = 0$, and is invariant under the 32 abelian fermionic isometries

$$\delta\Theta^{rj} = \alpha^{rj}, \quad \delta\overline{\Theta}_{jr} = \overline{\alpha}_{jr} \tag{1.7}$$

where α^{rj} and $\overline{\alpha}_{jr}$ are constant Grassmann parameters. Note that the bosonic isometries of this model include a "bonus" U(1) symmetry [9] in addition to the SU(2, 2) $\times$ SU(4) isometries of the original $AdS_5 \times S^5$ sigma model.

Introducing fermionic worldsheet superfields whose bosonic components are twistor-like coordinates has been useful in classical descriptions of the superstring where kappa-symmetry is replaced by worldsheet supersymmetry [10 – 12]. The N=2 model in this paper shares many features with this "super-embedding" approach, however, it has the advantage of being quantizable because of the second-order action for the fermionic superfields. Since the second-order action for fermionic superfields is generated by the Ramond-Ramond background, it might be possible to generalize the twistor-like methods of this paper to more general Ramond-Ramond backgrounds.

The abelianization of the fermionic isometries of (1.7) comes from setting $T_{\alpha\beta}{}^a = 0$ and means that the supersymmetry generators anticommute with each other. To relate this model to super-Yang-Mills where supersymmetry acts in the conventional way, it is useful to interpret (1.5) as the limit of a non-linear topological A-model which is constructed such that the isometries of (1.6) and (1.7) are deformed into SU$(2,2|4)$ isometries.

The worldsheet action for this non-linear topological A-model is

$$S = \frac{1}{\Lambda}\int d^2z d^4\kappa \left[\overline{\Theta}_{rj}\Theta^{jr} - \frac{1}{2R^2}\overline{\Theta}_{rj}\Theta^{js}\overline{\Theta}_{sk}\Theta^{kr} + \frac{1}{3R^4}\overline{\Theta}_{rj}\Theta^{js}\overline{\Theta}_{sk}\Theta^{kt}\overline{\Theta}_{tl}\Theta^{lr} + \cdots\right] \quad (1.8)$$
$$= \frac{R^2}{\Lambda}\int d^2z d^4\kappa Tr\left[\log\left(1+\frac{1}{R^2}\overline{\Theta}\Theta\right)\right]$$

where R is a new parameter which, in the limit $R \to \infty$, takes the non-linear sigma model into the linear sigma model of (1.5). This non-linear action will be shown to be one-loop conformally invariant, and is invariant under the same SU$(2,2)\times$ SU$(4)\times$ U(1) transformations as (1.6). But the fermionic transformations of (1.7) are modified to

$$\delta\Theta^{rj} = \alpha^{rj} + \frac{1}{R^2}\Theta^{rk}\overline{\alpha}_{ks}\Theta^{sj}, \quad \delta\overline{\Theta}_{jr} = \overline{\alpha}_{jr} + \frac{1}{R^2}\overline{\Theta}_{js}\alpha^{sk}\overline{\Theta}_{kr}, \quad (1.9)$$

which anticommute to form the superalgebra SU$(2,2|4)$.

It will be conjectured that the BRST cohomology in the closed string sector of this non-linear topological A-model is trivial, which implies that the open string physical states are independent of R and Λ in (1.8). This would be similar to the topological A-model for d=3 Chern-Simons which has physical states only in the open string sector [13], but would be different from the topological B-model for the twistor-string [14] which describes N=4 d=4 super-Yang-Mills in the open sector and N=4 d=4 conformal supergravity in the closed sector.

In the topological A-model for d=3 Chern-Simons, the open string boundary conditions are $X^I = \overline{X}_I$ where X^I and $\overline{X}_I$ are chiral and anti-chiral superfields for $I = 1$ to 3. Similarly, the open string boundary conditions in the non-linear topological A-model of (1.8) are $\Theta^{rj} = \overline{\Theta}_{jr}$. These boundary conditions eliminate half of the 32 θ's and break SU$(2,2|4)$ invariance down to an OSp$(4|4)$ subgroup, which is the N=4 supersymmetry algebra on AdS_4. In this open topological A-model, the BRST cohomology of physical states will be shown to describe d=4 N=4 super-Yang-Mills, where the bosonic components of Θ^{rj} are interpreted as twistor coordinates constructed from the four x's of AdS_4 together with an N=4 d=4 pure spinor.

The similarities between Chern-Simons and N=4 d=4 super-Yang-Mills are not surprising since, using the pure spinor formalism, the d=10 super-Yang-Mills action can be written in the Chern-Simons form $S = \langle VQV + \frac{2}{3}V^3\rangle$ where Q is the pure spinor BRST operator and V is the super-Yang-Mills vertex operator [15, 16]. Furthermore, there is a gauge/geometry correspondence relating Chern-Simons and the resolved conifold which has many features in common with the Maldacena conjecture relating N=4 d=4 super-Yang-Mills and $AdS_5 \times S^5$. The Chern-Simons/conifold correspondence was first proposed by Gopakumar and Vafa [17], and was later proven using open-closed duality arguments by Ooguri and Vafa [18].

The basic idea behind the open-closed duality proof of Gopakumar-Vafa-Ooguri is that, in a certain limit, the closed topological string theory for the resolved conifold geometry develops a new branch corresponding to "holes" on the closed worldsheet. These holes were then shown to correspond to the open string sector of the topological A-model that describes d=3 Chern-Simons.

Since the open string sector of the topological A-model in this paper describes d=4 N=4 super-Yang-Mills, and since this topological A-model is related to a certain limit of the closed superstring in an $AdS_5 \times S^5$ background, it is natural to try to construct a similar open-closed duality proof for the Maldacena conjecture. However, there are some questions that need to be answered before such a proof can be attempted.

One question is to explain the interpretation of the torsion ratio of (1.3) as the $AdS \times S^5$ radius. Although this interpretation is easily understood in the flat space limit where $r \to \infty$, it is not obvious this interpretation is correct in the limit where $r \to 0$. So it is not clear that the limit discussed in this paper corresponds to weak coupling on the super-Yang-Mills side of the duality.

A second question is to compute the complete cohomology of physical states for the topological A-model of (1.8). Although it will be shown that the cohomology in the open string sector of this A-model describes d=4 N=4 super-Yang-Mills, it remains to be shown that there are no physical states in the closed string sector of this A-model.

Finally, a third question which needs to be answered is if the open string topological A-model in this paper can be interpreted as a branch of the closed string $AdS_5 \times S^5$ sigma model which emerges in the limit where $T_{\alpha\beta}{}^a \to 0$. Perhaps the "bonus" U(1) symmetry in (1.6) will play a role in the emergence of this branch.

In section 2 of this paper, the $AdS_5 \times S^5$ sigma model using the pure spinor formalism is reviewed and the flat space limit is discussed. In section 3, the $AdS_5 \times S^5$ sigma model is shown to reduce to a linear topological A-model in the limit where $T_{\alpha\beta}{}^a \to 0$. In section 4, this linear topological A-model is deformed into a non-linear topological A-model with $PSU(2,2|4)$ invariance. And in section 5, the open string sector of this non-linear topological A-model is shown to describe d=4 N=4 super-Yang-Mills.

2. Review of pure spinor formalism in $AdS_5 \times S^5$ background

Using the pure spinor formalism, the superstring can be quantized in any consistent d=10 supergravity background [19]. Unlike the Green-Schwarz formalism where the gauge-fixing procedure of kappa-symmetry is poorly understood even in a flat background, the pure spinor formalism is quantized using a BRST operator which can be defined in any consistent supergravity background. In an $AdS_5 \times S^5$ background, the BRST transformations act in a geometric manner, which has been useful for proving the quantum consistency of this background [5].

2.1 Sigma model action

The sigma model for the superstring in an $AdS_5 \times S^5$ background is manifestly $PSU(2,2|4)$-

invariant and is constructed from the Metsaev-Tseytlin left-invariant currents [6]

$$J^A = (G^{-1}\partial G)^A, \quad \overline{J}^A = (G^{-1}\overline{\partial} G)^A, \tag{2.1}$$

where $G(x,\theta,\widehat{\theta})$ takes values in the coset $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$, $A = ([ab], c, \alpha, \widehat{\alpha})$ ranges over the 30 bosonic and 32 fermionic elements in the Lie algebra of $PSU(2,2|4)$, $[ab]$ labels the $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ "Lorentz" generators, $c = 0$ to 9 labels the "translation" generators, and $\alpha, \widehat{\alpha} = 1$ to 16 label the fermionic "supersymmetry" generators.

Although the $AdS_5 \times S^5$ background only preserves an $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ subgroup of $\mathrm{SO}(9,1)$ Lorentz-invariance, it will sometimes be convenient to use $\mathrm{SO}(9,1)$ 16-component notation for the spinor indices. Throughout this paper, both α and $\widehat{\alpha}$ labels a 16-component Majorana-Weyl spinor index when it is a superscript, and labels a 16-component Majorana-antiWeyl spinor index when it is a subscript. Even though α and $\widehat{\alpha}$ label spinors of the same ten-dimensional spacetime chirality, it will be convenient to use two types of indices where unhatted indices are associated with spinors coming from the left-moving sector of the Type IIB superstring and hatted indices are associated with spinors coming from the right-moving sector.

As in a flat background, the matrices $\gamma^c_{\alpha\beta}$ and $(\gamma^c)^{\alpha\beta}$ matrices are 16×16 symmetric matrices which form the off-diagonal blocks of the 32×32 ten-dimensional Γ-matrices, and which satisfy the anticommutation relation $\gamma^c_{\alpha\beta}(\gamma^d)^{\beta\gamma} + \gamma^d_{\alpha\beta}(\gamma^c)^{\beta\gamma} = 2\eta^{cd}\delta^\gamma_\alpha$. The matrices $\gamma^{[c_1\ldots c_N]}$ are constructed in the usual way by multiplying products of γ^c, e.g. $(\gamma^{[cd]})_\alpha{}^\gamma = \gamma^{[c}_{\alpha\beta}(\gamma^{d]})^{\beta\gamma}$, and satisfy the property that $\gamma^{c_1c_2c_3}_{\alpha\beta} = -\gamma^{c_1c_2c_3}_{\beta\alpha}$ and $\gamma^{c_1c_2c_3c_4c_5}_{\alpha\beta} = \gamma^{c_1c_2c_3c_4c_5}_{\beta\alpha}$. The five-form $\gamma^{01234}_{\alpha\widehat{\beta}}$ which is in the direction of the Ramond-Ramond flux will be denoted as $\eta_{\alpha\widehat{\beta}}$.

Under $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$, a 16-component spinor f^α decomposes into $f^{r'j'}$ where $r' = 1$ to 4 is an $\mathrm{SO}(4,1)$ spinor index and $j' = 1$ to 4 is an $\mathrm{SO}(5)$ spinor index. (Note that r' and j' indices can be raised and lowered in an $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ invariant manner.) If one expresses $J^A = (G^{-1}\partial G)^A$ as an 8×8 matrix which takes values in the Lie-algebra of $PSU(2,2|4)$, the upper right-hand off-diagonal 4×4 block $J^{r'}_{j'}$ is obtained from the $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ decomposition of the 16-component spinor $J^\alpha + iJ^{\widehat{\alpha}}$, whereas the lower left-hand off-diagonal 4×4 block $J^{j'}_{r'}$ is obtained from the $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ decomposition of the 16-component spinor $J^\alpha - iJ^{\widehat{\alpha}}$.

The action in the pure spinor formalism involves left and right-moving bosonic ghosts, $(\lambda^\alpha, w_\alpha)$ and $(\widehat{\lambda}^{\widehat{\alpha}}, \widehat{w}_{\widehat{\alpha}})$, which satisfy the pure spinor constraints $\lambda\gamma^c\lambda = \widehat{\lambda}\gamma^c\widehat{\lambda} = 0$. Because of the pure spinor constraints, w_α and $\widehat{w}_{\widehat{\alpha}}$ can only appear in combinations which are invariant under the gauge transformations

$$\delta w_\alpha = \xi^c(\gamma_c\lambda)_\alpha, \quad \delta\widehat{w}_{\widehat{\alpha}} = \widehat{\xi}^c(\gamma_c\widehat{\lambda})_{\widehat{\alpha}}. \tag{2.2}$$

As in standard coset constructions, the $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ coset $G(x,\theta,\widehat{\theta})$ is defined up to right multiplication by a local $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ parameter $\Omega^{[ab]}(x,\theta,\widehat{\theta})$ as

$$\delta G(x,\theta,\widehat{\theta}) = G(x,\theta,\widehat{\theta})\ (\Omega^{[ab]}(x,\theta,\widehat{\theta})T_{[ab]}) \tag{2.3}$$

where $T_{[ab]}$ are the $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ generators. Under these gauge transformations, the pure spinors are defined to transform covariantly as

$$\delta\lambda^\alpha = -\frac{1}{2}\Omega^{[ab]}(\gamma_{[ab]}\lambda)^\alpha, \quad \delta w_\alpha = \frac{1}{2}\Omega^{[ab]}(\gamma_{[ab]}w)_\alpha, \tag{2.4}$$
$$\delta\widehat{\lambda}^{\widehat{\alpha}} = -\frac{1}{2}\Omega^{[ab]}(\gamma_{[ab]}\lambda)^{\widehat{\alpha}}, \quad \delta\widehat{w}_{\widehat{\alpha}} = \frac{1}{2}\Omega^{[ab]}(\gamma_{[ab]}\widehat{w})_{\widehat{\alpha}}.$$

A convenient way to write the sigma model action in a manifestly gauge-invariant manner is [20, 2]

$$\begin{aligned} S = \frac{1}{\Lambda}\int d^2z\Big[&\frac{1}{2}\eta_{AB}(J^A-\mathcal{A}^A)(\overline{J}^B-\overline{\mathcal{A}}^B) \\ &+\mathcal{B}+w_\alpha\Big(\overline{\partial}\lambda+\frac{1}{2}\overline{\mathcal{A}}^{[ab]}\gamma_{[ab]}\lambda\Big)^\alpha+\widehat{w}_{\widehat{\alpha}}\Big(\partial\widehat{\lambda}+\frac{1}{2}\mathcal{A}^{[ab]}\gamma_{[ab]}\widehat{\lambda}\Big)^{\widehat{\alpha}}\Big] \\ = \frac{1}{\Lambda}\int d^2z\Big[&\frac{1}{2}\eta_{[ab][cd]}(J^{[ab]}-\mathcal{A}^{[ab]})(\overline{J}^{[cd]}-\overline{\mathcal{A}}^{[cd]})+\frac{1}{2}\eta_{cd}J^c\overline{J}^d+\frac{1}{4}\eta_{\alpha\widehat{\beta}}(J^{\widehat{\beta}}\overline{J}^\alpha+\overline{J}^{\widehat{\beta}}J^\alpha) \\ &+\frac{1}{2}\eta_{\alpha\widehat{\beta}}(J^{\widehat{\beta}}\overline{J}^\alpha-\overline{J}^{\widehat{\beta}}J^\alpha)+w_\alpha\Big(\overline{\partial}\lambda+\frac{1}{2}\overline{\mathcal{A}}^{[ab]}\gamma_{[ab]}\lambda\Big)^\alpha+\widehat{w}_{\widehat{\alpha}}\Big(\partial\widehat{\lambda}+\frac{1}{2}\mathcal{A}^{[ab]}\gamma_{[ab]}\widehat{\lambda}\Big)^{\widehat{\alpha}}\Big], \end{aligned} \tag{2.5}$$

where η_{AB} is the $PSU(2,2|4)$ metric, $\eta_{[ab][cd]}=\eta_{a[c}\eta_{d]b}$ when $a,b,c,d=0$ to 4, $\eta_{[ab][cd]}=-\eta_{a[c}\eta_{d]b}$ when $a,b,c,d=5$ to 9, η_{cd} is the d=10 Minkowski metric, $\eta_{\alpha\widehat{\beta}}=(\gamma^{01234})_{\alpha\widehat{\beta}}$, $\mathcal{A}^{[ab]}$ and $\overline{\mathcal{A}}^{[ab]}$ are worldsheet $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ gauge fields, and $\mathcal{B}$ is the Wess-Zumino term which in an $AdS_5\times S^5$ background takes the simple form [20]

$$\mathcal{B}=\frac{1}{2}\eta_{\alpha\widehat{\beta}}(J^{\widehat{\beta}}\overline{J}^\alpha-\overline{J}^{\widehat{\beta}}J^\alpha). \tag{2.6}$$

Since $\mathcal{A}^{[ab]}$ and $\overline{\mathcal{A}}^{[ab]}$ satisfy auxiliary equations of motion, they can be integrated out to obtain the action

$$\begin{aligned} S = \frac{1}{\Lambda}\int d^2z\Big[&\frac{1}{2}\eta_{cd}J^c\overline{J}^d+\eta_{\alpha\widehat{\beta}}\Big(\frac{3}{4}J^{\widehat{\beta}}\overline{J}^\alpha-\frac{1}{4}\overline{J}^{\widehat{\beta}}J^\alpha\Big) \\ &+w_\alpha(\overline{\nabla}\lambda)^\alpha+\widehat{w}_{\widehat{\alpha}}(\nabla\widehat{\lambda})^{\widehat{\alpha}}-\frac{1}{2}\eta_{[ab][cd]}(w\gamma^{[ab]}\lambda)(\widehat{w}\gamma^{[cd]}\widehat{\lambda})\Big], \end{aligned} \tag{2.7}$$

where $(\overline{\nabla}\lambda)^\alpha=\overline{\partial}\lambda^\alpha+\frac{1}{2}\overline{J}^{[ab]}(\gamma_{[ab]}\lambda)^\alpha$ and $(\nabla\widehat{\lambda})^{\widehat{\alpha}}=\partial\widehat{\lambda}^{\widehat{\alpha}}+\frac{1}{2}J^{[ab]}(\gamma_{[ab]}\widehat{\lambda})^{\widehat{\alpha}}$. Using the Maurer-Cartan equations, the action of (2.7) can be shown to be invariant under the BRST transformation generated by [3]

$$Q+\overline{Q}=\int dz\ \eta_{\alpha\widehat{\alpha}}\lambda^\alpha J^{\widehat{\alpha}}+\int d\overline{z}\eta_{\alpha\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}\overline{J}^\alpha \tag{2.8}$$

which transform the $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ coset and pure spinor ghosts as

$$\delta G=G(\epsilon\lambda^\alpha T_\alpha+\epsilon\widehat{\lambda}^{\widehat{\alpha}}T_{\widehat{\alpha}}), \quad \delta w_\alpha=\epsilon\eta_{\alpha\widehat{\beta}}J^{\widehat{\beta}}, \quad \delta\widehat{w}_{\widehat{\alpha}}=\epsilon\eta_{\alpha\widehat{\beta}}\overline{J}^{\widehat{\beta}}, \tag{2.9}$$

where T_α and $T_{\widehat{\alpha}}$ are the 32 fermionic generators of $PSU(2,2|4)$ and ϵ is a constant Grassmann parameter.

This BRST invariance, together with $PSU(2,2|4)$ invariance, fixes the relative coefficients of the terms in the sigma model action of (2.7). So, naively, the $AdS_5 \times S^5$ radius r can only appear in the action through the coupling constant $\Lambda = \alpha'/r^2$. However, if one allows the $PSU(2,2|4)$ algebra to be deformed as the value of r is changed, the r dependence of the action can be more complicated and the form of the action can be modified. For example, in the flat space limit where $r \to \infty$, the $PSU(2,2|4)$ algebra is deformed to the N=2 d=10 super-Poincaré algebra. As will now be discussed, this modifies the sigma model action of (2.7) to a quadratic action.

2.2 Flat space limit

Although the naive limit as $r \to \infty$ is obtained by simply taking $\Lambda \to 0$ in the sigma model action of (2.7), this limit would preserve $PSU(2,2|4)$ invariance instead of the desired N=2 d=10 super-Poincaré invariance of flat Minkowski superspace. To obtain the correct flat space limit, one needs to rescale the $PSU(2,2|4)$ structure constants such that when $r \to \infty$, the $PSU(2,2|4)$ algebra is deformed into the N=2 d=10 super-Poincaré algebra.

The non-vanishing $PSU(2,2|4)$ structure constants f_{AB}^C are

$$\begin{aligned}
f_{\alpha\beta}^{c} &= \gamma_{\alpha\beta}^{c}, \qquad f_{\widehat{\alpha}\widehat{\beta}}^{c} = \gamma_{\alpha\beta}^{c}, \\
f_{\alpha c}^{\widehat{\beta}} &= -\gamma_{c\alpha\beta}\eta^{\beta\widehat{\beta}}, \qquad f_{\widehat{\alpha} c}^{\beta} = -\gamma_{c\widehat{\alpha}\widehat{\beta}}\eta^{\beta\widehat{\beta}}, \\
f_{\alpha\widehat{\beta}}^{[ef]} &= \pm(\gamma^{ef})_{\alpha}{}^{\gamma}\eta_{\gamma\widehat{\beta}}, \quad f_{cd}^{[ef]} = \pm\delta_c^{[e}\delta_d^{f]}, \\
f_{[cd][ef]}^{[gh]} &= \eta_{ce}\delta_d^{[g}\delta_f^{h]} - \eta_{cf}\delta_d^{[g}\delta_e^{h]} + \eta_{df}\delta_c^{[g}\delta_e^{h]} - \eta_{de}\delta_c^{[g}\delta_f^{h]}, \\
f_{[cd]e}^{f} &= \eta_{e[c}\delta_{d]}^{f}, \qquad f_{[cd]\alpha}^{\beta} = \frac{1}{2}(\gamma_{cd})_{\alpha}{}^{\beta}, \quad f_{[cd]\widehat{\alpha}}^{\widehat{\beta}} = \frac{1}{2}(\gamma_{cd})_{\widehat{\alpha}}{}^{\widehat{\beta}},
\end{aligned} \tag{2.10}$$

where the + sign in the third line is if $(c,d,e,f) = 0$ to 4, and the − sign is if $(c,d,e,f) = 5$ to 9.

To deform these structure constants to the super-Poincaré structure constants in the $r \to \infty$ limit, one should rescale (2.10) such that

$$\begin{aligned}
f_{\alpha\beta}^{c} &= \gamma_{\alpha\beta}^{c}, \qquad f_{\widehat{\alpha}\widehat{\beta}}^{c} = \gamma_{\alpha\beta}^{c}, \\
f_{\alpha c}^{\widehat{\beta}} &= -r^{-1}\gamma_{c\alpha\beta}\eta^{\beta\widehat{\beta}}, \qquad f_{\widehat{\alpha} c}^{\beta} = -r^{-1}\gamma_{c\widehat{\alpha}\widehat{\beta}}\eta^{\beta\widehat{\beta}}, \\
f_{\alpha\widehat{\beta}}^{[ef]} &= \pm r^{-2}(\gamma^{ef})_{\alpha}{}^{\gamma}\eta_{\gamma\widehat{\beta}}, \quad f_{cd}^{[ef]} = \pm r^{-2}\delta_c^{[e}\delta_d^{f]}, \\
f_{[cd][ef]}^{[gh]} &= \eta_{ce}\delta_d^{[g}\delta_f^{h]} - \eta_{cf}\delta_d^{[g}\delta_e^{h]} + \eta_{df}\delta_c^{[g}\delta_e^{h]} - \eta_{de}\delta_c^{[g}\delta_f^{h]} \\
f_{[cd]e}^{f} &= \eta_{e[c}\delta_{d]}^{f}, \qquad f_{[cd]\alpha}^{\beta} = \frac{1}{2}(\gamma_{cd})_{\alpha}{}^{\beta}, \quad f_{[cd]\widehat{\alpha}}^{\widehat{\beta}} = \frac{1}{2}(\gamma_{cd})_{\widehat{\alpha}}{}^{\widehat{\beta}}.
\end{aligned} \tag{2.11}$$

The metric g_{AB} should satisfy the property that $f_{AB}^C\, g_{CD}$ is graded-antisymmetric under permutations of $[ABD]$, so the rescaling of (2.11) implies one should also rescale $g_{\alpha\widehat{\beta}} = \eta_{\alpha\widehat{\beta}}$ and $g_{[ab][cd]} = \eta_{[ab][cd]}$ to

$$g_{\alpha\widehat{\beta}} = r\eta_{\alpha\widehat{\beta}}, \quad g_{[ab][cd]} = r^2\eta_{[ab][cd]}. \tag{2.12}$$

JHEP08(2007)011

Since the structure constants f_{AB}^{C} are proportional to the superspace torsions $T_{AB}{}^{C}$, the rescaling of (2.11) implies that

$$\frac{T_{\alpha\beta}{}^{b}\eta_{ab}}{T_{\alpha a}{}^{\widehat{\beta}}\eta_{\beta\widehat{\beta}}} = r. \tag{2.13}$$

If $T_{\alpha\beta}{}^{b}$ is fixed to satisfy $T_{\alpha\beta}{}^{b} = \gamma^{b}_{\alpha\beta}$, (2.13) implies that $T_{\alpha c}{}^{\widehat{\beta}} = r^{-1}\gamma_{c\alpha\beta}\eta^{\beta\widehat{\beta}}$, which is the correct r dependence since the AdS curvature $R_{ab\alpha}{}^{\beta}$ goes like $1/r^2$, and Bianchi identities imply that $R_{ab\alpha}{}^{\beta}$ is proportional to $T_{a\alpha}{}^{\gamma}T_{b\gamma}{}^{\beta}$.

Since $g_{\alpha\widehat{\beta}} = r\eta_{\alpha\widehat{\beta}}$ blows up when $r \to \infty$, it is convenient to write the second-order kinetic term for the fermions in (2.7) in the first-order form as

$$\begin{aligned}
&\frac{1}{\Lambda}\int d^2z r\eta_{\alpha\widehat{\beta}}\left(\frac{3}{4}J^{\widehat{\beta}}\overline{J}^{\alpha} - \frac{1}{4}\overline{J}^{\widehat{\beta}}J^{\alpha}\right) \qquad (2.14)\\
&= \frac{1}{\Lambda}\int d^2z r\eta_{\alpha\widehat{\beta}}\left(\frac{1}{2}J^{\widehat{\beta}}\overline{J}^{\alpha} + \frac{1}{4}J^{\widehat{\beta}} \wedge J^{\alpha}\right)\\
&= \frac{1}{\Lambda}\int d^2z\left[\overline{J}^{\alpha}d_{\alpha} + J^{\widehat{\alpha}}\widehat{d}_{\widehat{\alpha}} + 2r^{-1}\eta^{\alpha\widehat{\beta}}d_{\alpha}\widehat{d}_{\widehat{\beta}} + \frac{1}{4}r\eta_{\alpha\widehat{\beta}}\int d\sigma_3\ d(J^{\widehat{\beta}} \wedge J^{\alpha})\right]\\
&= \frac{1}{\Lambda}\int d^2z\left[\overline{J}^{\alpha}d_{\alpha} + J^{\widehat{\alpha}}\widehat{d}_{\widehat{\alpha}} + 2r^{-1}\eta^{\alpha\widehat{\beta}}d_{\alpha}\widehat{d}_{\widehat{\beta}} + \frac{1}{4}\int d\sigma_3\left(\gamma_{c\alpha\beta}J^{c}\wedge J^{\alpha}\wedge J^{\beta} - \gamma_{c\widehat{\alpha}\widehat{\beta}}J^{c}\wedge J^{\widehat{\alpha}}\wedge J^{\widehat{\beta}}\right)\right]
\end{aligned}$$

where d_{α} and $\widehat{d}_{\widehat{\alpha}}$ are auxiliary variables and the two-form $J^{\widehat{\beta}} \wedge J^{\alpha} \equiv J^{\widehat{\beta}}\overline{J}^{\alpha} - \overline{J}^{\widehat{\beta}}J^{\alpha}$ has been written as the integral of a Wess-Zumino-Witten three-form using the Maurer-Cartan equations

$$dJ^{\widehat{\beta}} = f^{\widehat{\beta}}_{c\alpha}J^{c} \wedge J^{\alpha} = r^{-1}\gamma_{c\alpha\beta}\eta^{\beta\widehat{\beta}}J^{c} \wedge J^{\alpha}, \tag{2.15}$$

$$dJ^{\beta} = f^{\beta}_{c\widehat{\alpha}}J^{c} \wedge J^{\widehat{\alpha}} = r^{-1}\gamma_{c\widehat{\alpha}\widehat{\beta}}\eta^{\beta\widehat{\beta}}J^{c} \wedge J^{\widehat{\alpha}}. \tag{2.16}$$

Furthermore, the BRST operator $Q + \overline{Q}$ of (2.8) can be written as

$$Q + \overline{Q} = \int dz\lambda^{\alpha}d_{\alpha} + \int d\overline{z}\widehat{\lambda}^{\widehat{\alpha}}\widehat{d}_{\widehat{\alpha}} \tag{2.17}$$

using the auxiliary equations of motion for d_{α} and $\widehat{d}_{\widehat{\alpha}}$.

When $r = \infty$, the left-invariant currents $(J^{c}, J^{\alpha}, J^{\widehat{\beta}}, J^{[ab]})$ simplify to

$$J^{c} = \Pi^{c} = \partial x^{c} + \theta\gamma^{c}\partial\theta + \widehat{\theta}\gamma^{c}\partial\widehat{\theta}, \quad J^{\alpha} = \partial\theta^{\alpha}, \quad J^{\widehat{\beta}} = \partial\widehat{\theta}^{\widehat{\beta}}, \quad J^{[ab]} = 0. \tag{2.18}$$

So the action of (2.7) reduces to

$$\begin{aligned}
S = \frac{1}{\Lambda}\int d^2z\Big[&\frac{1}{2}\eta_{cd}\Pi^{c}\overline{\Pi}^{d} - d_{\alpha}\overline{\partial}\theta^{\alpha} - \widehat{d}_{\widehat{\alpha}}\partial\widehat{\theta}^{\widehat{\alpha}} + w_{\alpha}\overline{\partial}\lambda^{\alpha} + \widehat{w}_{\widehat{\alpha}}\partial\widehat{\lambda}^{\widehat{\alpha}}\\
&+\frac{1}{4}\int d\sigma_3(\gamma_{c\alpha\beta}\Pi^{c} \wedge \partial\theta^{\alpha} \wedge \partial\theta^{\beta} - \gamma_{c\widehat{\alpha}\widehat{\beta}}\Pi^{c} \wedge \partial\widehat{\theta}^{\widehat{\alpha}} \wedge \partial\widehat{\theta}^{\widehat{\beta}})\Big],
\end{aligned}$$

which is the worldsheet action in a flat background using the pure spinor formalism. By defining

$$p_{\alpha} = d_{\alpha} + \cdots, \quad \widehat{p}_{\widehat{\alpha}} = \widehat{d}_{\widehat{\alpha}} + \cdots \tag{2.19}$$

where $\dots$ are functions of $(x, \theta, \widehat{\theta})$, this action can be written in quadratic form as [2]

$$S = \frac{1}{\Lambda}\int d^2z\left[\frac{1}{2}\eta_{cd}\partial x^c\overline{\partial}x^d - p_\alpha\overline{\partial}\theta^\alpha - \widehat{p}_{\widehat{\alpha}}\partial\widehat{\theta^{\widehat{\alpha}}} + w_\alpha\overline{\partial}\lambda^\alpha + \widehat{w}_{\widehat{\alpha}}\partial\widehat{\lambda^{\widehat{\alpha}}}\right]. \tag{2.20}$$

3. New limit of sigma model

In the previous section, we constructed the flat space limit of the $AdS_5 \times S^5$ sigma model in which $T_{c\alpha}{}^{\widehat{\beta}} \to 0$ and $T_{\alpha\beta}{}^c = \gamma^c_{\alpha\beta}$. In this section, we shall consider a different limit of the model in which $T_{\alpha\beta}{}^c \to 0$ and $T_{c\alpha}{}^{\widehat{\beta}} = \gamma_{c\alpha\beta}\eta^{\beta\widehat{\beta}}$. If one defines r as in (2.13), this formally corresponds to the limit $r \to 0$ of the $AdS_5 \times S^5$ background. However, since supergravity backgrounds are usually defined such that $T_{\alpha\beta}{}^c = \gamma^c_{\alpha\beta}$ [7], this limit cannot be identified with a conventional supergravity background.

3.1 $T_{\alpha\beta}{}^c \to 0$ limit

To construct the sigma model in this new limit, one needs to rescale the $PSU(2,2|4)$ structure constants of (2.10) as

$$\begin{aligned}
f^c_{\alpha\beta} &= r\gamma^c_{\alpha\beta}, & f^c_{\widehat{\alpha}\widehat{\beta}} &= r\gamma^c_{\alpha\beta}, \\
f^{\widehat{\beta}}_{\alpha c} &= -\gamma_{c\alpha\beta}\eta^{\beta\widehat{\beta}}, & f^{\beta}_{\widehat{\alpha} c} &= -\gamma_{c\widehat{\alpha}\widehat{\beta}}\eta^{\beta\widehat{\beta}}, \\
f^{[ef]}_{\alpha\widehat{\beta}} &= \pm r(\gamma^{ef})_\alpha{}^\gamma\eta_{\gamma\widehat{\beta}}, & f^{[ef]}_{cd} &= \pm\delta^{[e}_c\delta^{f]}_d, \\
f^{[gh]}_{[cd][ef]} &= \eta_{ce}\delta^{[g}_d\delta^{h]}_f - \eta_{cf}\delta^{[g}_d\delta^{h]}_e + \eta_{df}\delta^{[g}_c\delta^{h]}_e - \eta_{de}\delta^{[g}_c\delta^{h]}_f \\
f^f_{[cd]e} &= \eta_{e[c}\delta^f_{d]}, & f^\beta_{[cd]\alpha} &= \frac{1}{2}(\gamma_{cd})_\alpha{}^\beta, \quad f^{\widehat{\beta}}_{[cd]\widehat{\alpha}} = \frac{1}{2}(\gamma_{cd})_{\widehat{\alpha}}{}^{\widehat{\beta}}.
\end{aligned} \tag{3.1}$$

Furthermore, to preserve the graded-antisymmetry of $f^C_{AB}\, g_{CD}$ under permutation of $[ABD]$, one needs to also rescale $g_{ab} = \eta_{ab}$ and $g_{[ab][cd]} = \eta_{[ab][cd]}$ to

$$g_{ab} = r^{-1}\eta_{ab}, \quad g_{[ab][cd]} = r^{-1}\eta_{[ab][cd]}. \tag{3.2}$$

When $r \to 0$, the structure constants $f^A_{\alpha\beta} \to 0$ which implies that the 32 fermionic isometries become abelian. In this limit, the $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ coset G splits into a bosonic coset $H^r_{r'}$ for $r, r' = 1$ to 4 which parameterizes $AdS_5 = \frac{\mathrm{SU}(2,2)}{\mathrm{SO}(4,1)}$, a bosonic coset $\widetilde{H}^j_{j'}$ for $j, j' = 1$ to 4 which parameterizes $S^5 = \frac{\mathrm{SU}(4)}{\mathrm{SO}(5)}$, and two fermionic matrices θ^{rj} and $\overline{\theta}_{jr}$ for $r, j = 1$ to 4. The index $r = 1$ to 4 labels a fundamental representation of the global $\mathrm{SU}(2,2)$, and the index $j = 1$ to 4 labels a fundamental representation of the global $\mathrm{SU}(4)$. Furthermore, the index $r' = 1$ to 4 labels a spinor representation of the local $\mathrm{SO}(4,1)$, and the index $j' = 1$ to 4 labels a spinor representation of of the local $\mathrm{SO}(5)$. Note that r' indices can be raised and lowered with an antisymmetric $\mathrm{SO}(4,1)$-invariant tensor $\epsilon^{r's'}$, and j' indices can be raised and lowered with an antisymmetric $\mathrm{SO}(5)$-invariant tensor $\epsilon^{j'k'}$. Under the 32 global fermionic isometries,

$$\delta\theta^{rj} = \alpha^{rj}, \quad \delta\overline{\theta}_{jr} = \overline{\alpha}_{jr}, \quad \delta H^r_{r'} = 0, \quad \delta\widetilde{H}^j_{j'} = 0, \tag{3.3}$$

where α^{rj} and α_{jr} are constant Grassmann parameters.

Since $g_{ab} = r^{-1}\eta_{ab}$ blows up when $r \to 0$, it is convenient to write the second-order kinetic term for the bosons in the first-order form as

$$\frac{1}{2\Lambda}\int d^2z\Big[r^{-1}\eta_{[ab][cd]}(J^{[ab]}-\mathcal{A}^{[ab]})(\overline{J}^{[cd]}-\overline{\mathcal{A}}^{[cd]})+r^{-1}\eta_{cd}J^c\overline{J}^d\Big] \tag{3.4}$$
$$=\frac{1}{\Lambda}\int d^2z\Big[(J^{[ab]}-\mathcal{A}^{[ab]})\overline{P}_{[ab]}+(\overline{J}^{[ab]}-\overline{\mathcal{A}}^{[ab]})P_{[ab]}+J^c\overline{P}_c+\overline{J}^cP^c$$
$$+2r(\eta^{[ab][cd]}P_{[ab]}\overline{P}_{[cd]}+\eta^{cd}P_c\overline{P}_d)\Big]$$

where $[P_{[ab]}, \overline{P}_{[ab]}, P_c, \overline{P}_c]$ are auxiliary fields. So the $AdS_5\times S^5$ sigma model action of (2.5) reduces in this limit $r\to 0$ to

$$S=\frac{1}{\Lambda}\int d^2z\Big[(J^{[ab]}-\mathcal{A}^{[ab]})\overline{P}_{[ab]}+(\overline{J}^{[ab]}-\overline{\mathcal{A}}^{[ab]})P_{[ab]}+J^c\overline{P}_c+\overline{J}^cP^c \tag{3.5}$$
$$+\frac{1}{4}\eta_{\alpha\widehat{\beta}}(J^{\widehat{\beta}}\overline{J}^{\alpha}+\overline{J}^{\widehat{\beta}}J^{\alpha})+\mathcal{B}+w_\alpha\left(\overline{\partial}\lambda+\frac{1}{2}\overline{\mathcal{A}}^{[ab]}\gamma_{[ab]}\lambda\right)^{\alpha}+\widehat{w}_{\widehat{\alpha}}\left(\partial\widehat{\lambda}+\frac{1}{2}\mathcal{A}^{[ab]}\gamma_{[ab]}\widehat{\lambda}\right)^{\widehat{\alpha}}\Big]$$

where $\mathcal{B}$ is the Wess-Zumino-Witten term of (2.6). Since $\int d^2z\mathcal{B} = \frac{1}{2}\int d^2z\int d\sigma_3(\gamma_{c\alpha\beta}J^c\wedge J^\alpha\wedge J^\beta - \gamma_{c\widehat{\alpha}\widehat{\beta}}J^c\wedge J^{\widehat{\alpha}}\wedge J^{\widehat{\beta}})$, the Wess-Zumino-Witten term can be eliminated from the action by shifting P_c and $\overline{P}_c$.

Furthermore, when $r\to 0$, the currents J^c and $J^{[cd]}$ simplify to

$$J^c = (H^{-1}\partial H)_{r'}^{s'}(\sigma^c)_{s'}^{r'},\quad J^{[cd]}=(H^{-1}\partial H)_{r'}^{s'}(\sigma^{[cd]})_{s'}^{r'}\quad \text{when}\quad c,d=0\ \text{to}\ 4, \tag{3.6}$$
$$J^c = (\widetilde{H}^{-1}\partial \widetilde{H})_{j'}^{k'}(\sigma^c)_{k'}^{j'},\quad J^{[cd]}=(\widetilde{H}^{-1}\partial \widetilde{H})_{j'}^{k'}(\sigma^{[cd]})_{k'}^{j'}\quad \text{when}\quad c,d=5\ \text{to}\ 9, \tag{3.7}$$

where σ^c and $\sigma^{[cd]}$ are 4×4 Pauli matrices which generate an SU(2, 2) algebra when $c = 0$ to 4, and generate an SU(4) algebra when $c = 5$ to 9. Expressing the SO(9, 1) spinors J^α and $J^{\widehat{\alpha}}$ in terms of SO(4, 1) × SO(5) spinors as $J^\alpha = J^{r'j'}$ and $J^{\widehat{\alpha}} = \widehat{J}^{r'j'}$, one finds that when $r\to 0$, $J^{r'j'}$ and $\widehat{J}^{r'j'}$ simplify to

$$J^{r'j'} = (H^{-1})_r^{r'}(\widetilde{H}^{-1})_j^{j'}\partial\theta^{rj}+\epsilon^{r's'}\epsilon^{j'k'}H_{s'}^r\widetilde{H}_{k'}^j\partial\overline{\theta}_{jr}, \tag{3.8}$$
$$\widehat{J}^{r'j'} = (H^{-1})_r^{r'}(\widetilde{H}^{-1})_j^{j'}\partial\theta^{rj}-\epsilon^{r's'}\epsilon^{j'k'}H_{s'}^r\widetilde{H}_{k'}^j\partial\overline{\theta}_{jr}.$$

Plugging these currents into (3.5), one finds that the action simplifies to

$$S=\frac{1}{\Lambda}\int d^2z\Big[(J^{[ab]}-\mathcal{A}^{[ab]})\overline{P}_{[ab]}+(\overline{J}^{[ab]}-\overline{\mathcal{A}}^{[ab]})P_{[ab]}+J^c\overline{P}_c+\overline{J}^cP^c \tag{3.9}$$
$$+\partial\overline{\theta}_{jr}\partial\theta^{rj}+w_\alpha\left(\overline{\partial}\lambda+\frac{1}{2}\overline{\mathcal{A}}^{[ab]}\gamma_{[ab]}\lambda\right)^{\alpha}+\widehat{w}_{\widehat{\alpha}}\left(\partial\widehat{\lambda}+\frac{1}{2}\mathcal{A}^{[ab]}\gamma_{[ab]}\widehat{\lambda}\right)^{\widehat{\alpha}}\Big].$$

3.2 Twistor-like variables

The final step in simplifying this action is to express the pure spinors in SO(4, 1) × SO(5) notation as $\lambda^\alpha = \lambda^{r'j'}$ and $\widehat{\lambda}^{\widehat{\alpha}} = \widehat{\lambda}^{r'j'}$ and to define the new variables Z^{rj} and $\overline{Z}_{jr}$ as

$$Z^{rj} = H_{r'}^r\widetilde{H}_{j'}^j\lambda^{r'j'},\quad \overline{Z}_{jr} = (H^{-1})_r^{r'}(\widetilde{H}^{-1})_j^{j'}\widehat{\lambda}_{j'r'} \tag{3.10}$$

where $\widehat{\lambda}_{j'r'} = \epsilon_{j'k'}\epsilon_{r's'}\widehat{\lambda}^{s'k'}$. Note that Z^{rj} and $\overline{Z}_{jr}$ are twistor-like variables since they transform covariantly under the global $SU(2,2) \times SU(4)$ isometries and since they are constructed out of the pure spinors and the ten x's parameterized by the cosets H and $\widetilde{H}$. Similarly, one can define the conjugate twistor-like variables Y_{jr} and $\overline{Y}^{rj}$ as

$$Y_{jr} = (H^{-1})_r^{r'}(\widetilde{H}^{-1})_j^{j'} w_{j'r'}, \quad \overline{Y}^{rj} = H^r_{r'}\widetilde{H}^j_{j'}\widehat{w}^{r'j'} \tag{3.11}$$

where $w_\alpha = w_{j'r'}$ and $\widehat{w}_{\widehat{\alpha}} = \epsilon_{j'k'}\epsilon_{r's'}\widehat{w}^{s'k'}$ are the original conjugate pure spinor variables written in $SO(4,1) \times SO(5)$ notation.

Using

$$Y_{jr}\overline{\partial} Z^{jr} = w_\alpha \overline{\partial}\lambda^\alpha + (H^{-1}\overline{\partial} H)^{r'}_{s'} w_{j'r'}\lambda^{s'j'} + (\widetilde{H}^{-1}\overline{\partial}\widetilde{H})^{j'}_{k'} w_{j'r'}\lambda^{r'k'}, \tag{3.12}$$

one finds that

$$w_\alpha\overline{\partial}\lambda^\alpha = Y_{jr}\overline{\partial}Z^{rj} - (w\sigma_c\lambda)\overline{J}^c - \frac{1}{2}(w\sigma_{[cd]}\lambda)\overline{J}^{[cd]} \tag{3.13}$$

where $(w\sigma_c\lambda) = w_{j'r'}(\sigma_c)^{r'}_{s'}\lambda^{s'j'}$ and $(w\sigma_{[cd]}\lambda) = w_{j'r'}(\sigma_{[cd]})^{r'}_{s'}\lambda^{s'j'}$ for $c = 0$ to 4, and $(w\sigma_c\lambda) = w_{j'r'}(\sigma_c)^{j'}_{k'}\lambda^{r'k'}$ and $(w\sigma_{[cd]}\lambda) = w_{j'r'}(\sigma_{[cd]})^{j'}_{k'}\lambda^{r'k'}$ for $c = 5$ to 9. Similarly,

$$\widehat{w}_{\widehat{\alpha}}\partial\widehat{\lambda}^{\widehat{\alpha}} = \overline{Y}^{rj}\partial\overline{Z}_{jr} - (\widehat{w}\sigma_c\widehat{\lambda})J^c - \frac{1}{2}(\widehat{w}\sigma_{[cd]}\widehat{\lambda})J^{[cd]}. \tag{3.14}$$

So after defining

$$\begin{aligned} P'^c &= P^c - (w\sigma^c\lambda), & \overline{P}'^c &= \overline{P}^c - (\widehat{w}\sigma^c\widehat{\lambda}), \\ P'^{[cd]} &= P^{[cd]} - \frac{1}{2}(w\sigma^{[cd]}\lambda), & \overline{P}'^{[cd]} &= \overline{P}^{[cd]} - \frac{1}{2}(\widehat{w}\sigma^{[cd]}\widehat{\lambda}), \end{aligned} \tag{3.15}$$

one can write the action of (3.9) as

$$\begin{aligned} S = \frac{1}{\Lambda}\int d^2z\Big[&(J^{[ab]} - \mathcal{A}^{[ab]})\overline{P}'_{[ab]} + (\overline{J}^{[ab]} - \overline{\mathcal{A}}^{[ab]})P'_{[ab]} + J^c\overline{P}'_c + \overline{J}^c P'^c \\ &+\partial\overline{\theta}_{jr}\partial\theta^{rj} + Y_{jr}\overline{\partial}Z^{rj} + \overline{Y}^{rj}\partial\overline{Z}_{jr}\Big]. \end{aligned} \tag{3.16}$$

The shift of (3.15) implies that under the gauge transformation $\delta w_\alpha = \xi^c(\gamma_c\lambda)_\alpha$ and $\delta\widehat{w}_{\widehat{\alpha}} = \widehat{\xi}^c(\gamma_c\widehat{\lambda})_{\widehat{\alpha}}$ of (2.2), P'_c and $\overline{P}'_c$ must transform as

$$\begin{aligned} \delta P'_c &= \xi^c\epsilon_{r's'}\epsilon_{j'k'}\lambda^{r'j}\lambda^{s'k'} = \xi^c(\lambda\gamma^{01234}\lambda), \\ \delta\overline{P}'_c &= \widehat{\xi}^c\epsilon^{r's'}\epsilon^{j'k'}\widehat{\lambda}_{r'j}\widehat{\lambda}_{s'k'} = \widehat{\xi}^c(\widehat{\lambda}\gamma^{01234}\widehat{\lambda}). \end{aligned} \tag{3.17}$$

So assuming that $(\lambda\gamma^{01234}\lambda)$ and $(\widehat{\lambda}\gamma^{01234}\widehat{\lambda})$ are non-zero, one can use this invariance to gauge-fix $P'^c = \overline{P}'^c = 0$. Furthermore, integrating out $\mathcal{A}^{[ab]}$ and $\overline{\mathcal{A}}^{[ab]}$ implies that $P'^{[ab]} = \overline{P}'^{[ab]} = 0$.

So finally, one can write the action in quadratic form as

$$S = \frac{1}{\Lambda}\int d^2z[\partial\overline{\theta}_{jr}\overline{\partial}\theta^{rj} + Y_{jr}\overline{\partial}Z^{rj} + \overline{Y}^{rj}\partial\overline{Z}_{jr}]. \tag{3.18}$$

Instead of the original action containing ten x's and 22 left and right-moving pure spinors, (3.18) contains 16 left-moving and 16 right-moving unconstrained bosonic spinors. So the second-order action for x has been converted into a first-order action for ten left and right-moving bosons which effectively removes the constraint on the pure spinors. The removal of the pure spinor constraint is related to the fact that $T_{\alpha\beta}{}^c = 0$ in this background. Since the BRST operator acts as $Q = \lambda^\alpha \nabla_\alpha$, $Q^2 = \lambda^\alpha \lambda^\beta \{\nabla_\alpha, \nabla_\beta\} = \lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^A \nabla_A$. When $T_{\alpha\beta}{}^c = \gamma^c_{\alpha\beta}$, the pure spinor constaint $\lambda\gamma^c\lambda = 0$ is required for Q to be nilpotent. However, when $T_{\alpha\beta}{}^c = 0$, the nilpotence of Q does not require λ^α to satisfy the pure spinor constraint.

3.3 $N = 2$ worldsheet supersymmetry

In terms of the variables $(\theta^{rj}, \overline{\theta}_{jr}, Z^{rj}, \overline{Z}_{jr}, Y_{jr}, \overline{Y}^{rj})$, the BRST transformations are

$$\delta\theta^{rj} = \epsilon Z^{rj}, \quad \delta\overline{\theta}_{jr} = \epsilon\overline{Z}_{jr}, \quad \delta Y_{jr} = \epsilon\partial\overline{\theta}_{rj}, \quad \delta\overline{Y}^{rj} = \epsilon\overline{\partial}\theta^{rj}, \tag{3.19}$$

which are generated by $Q + \overline{Q}$ where

$$Q = \int dz Z^{rj} \partial\overline{\theta}_{jr}, \quad \overline{Q} = \int d\overline{z}\, \overline{Z}_{jr} \overline{\partial}\theta^{rj}. \tag{3.20}$$

Unlike in a flat background where it is difficult to construct b and $\overline{b}$ ghosts satisfying $\{Q, b\} = T$ and $\{\overline{Q}, \overline{b}\} = \overline{T}$, it is easy to construct b and $\overline{b}$ ghosts in this background as

$$b = Y_{jr} \partial\theta^{rj}, \quad \overline{b} = \overline{Y}^{rj} \overline{\partial\theta}_{jr}, \tag{3.21}$$

where

$$T = \partial\theta^{rj} \partial\overline{\theta}_{jr} + Y_{jr} \partial Z^{rj}, \quad \overline{T} = \overline{\partial}\theta^{rj} \overline{\partial\theta}_{jr} + \overline{Y}^{rj} \overline{\partial Z}_{jr}. \tag{3.22}$$

Since Y_{jr} and $\overline{Y}^{jr}$ have conformal weight $(1, 0)$ and $(0, 1)$, the action of (3.18) has A-twisted N=(2,2) supersymmetry and can be interpreted as a topological A-model. This topological A-model can be expressed in N=(2,2) superspace by combining the component fields into the chiral and antichiral superfields

$$\begin{aligned} \Theta^{rj} &= \theta^{rj} + \kappa_+ Z^{rj} + \kappa_- \overline{Y}^{rj} + \kappa_+\kappa_- f^{rj}, \\ \overline{\Theta}_{jr} &= \overline{\theta}_{jr} + \overline{\kappa}_+ Y_{jr} + \overline{\kappa}_- \overline{Z}_{jr} + \overline{\kappa}_+\overline{\kappa}_- \overline{f}_{jr}, \end{aligned} \tag{3.23}$$

where $(\kappa_+, \overline{\kappa}_+)$ and $(\kappa_-, \overline{\kappa}_-)$ are the left and right-moving N=(2,2) Grassmann parameters, and $(f^{rj}, \overline{f}_{jr})$ are auxiliary fields.

In terms of Θ^{rj} and $\overline{\Theta}_{jr}$, the action of (3.18) is

$$S = \frac{1}{\Lambda} \int d^2 z \int d^4\kappa \overline{\Theta}_{jr} \Theta^{rj}, \tag{3.24}$$

and the global bosonic isometries act as

$$\delta\Theta^{rj} = i\Lambda^r_s \Theta^{sj} + i\Theta^{rk}\Omega^j_k + i\Sigma\Theta^{rj}, \quad \delta\overline{\Theta}_{jr} = -i\overline{\Theta}_{js}\Lambda^s_r - i\Omega^k_j \overline{\Theta}_{kr} - i\Sigma\overline{\Theta}_{jr}, \tag{3.25}$$

where $(\Lambda^r_s, \Omega^k_j, \Sigma)$ are constant parameters satisfying $\Lambda^r_r = \Omega^j_j = 0$. Note that in addition to the $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$ bosonic isometries, there is an additional "bonus" U(1) symmetry parameterized by Σ. Under the fermionic isometries of (3.3), the superfields transform as

$$\delta\Theta^{rj} = \alpha^{rj}, \quad \delta\overline{\Theta}_{jr} = \overline{\alpha}_{jr}. \tag{3.26}$$

4. Non-linear topological A-model

To compute the physical states of the linear topological A-model of (3.24), it will be useful to define a non-linear topological A-model which reduces to the linear model of (3.24) in a certain large-radius limit. In the non-linear model, the $\mathrm{SU}(2,2) \times \mathrm{SU}(4) \times \mathrm{U}(1)$ bosonic isometries will combine with the 32 fermionic isometries to form an $\mathrm{SU}(2,2|4)$ supergroup. Since this supergroup includes the $PSU(2,2|4)$ isometries of the $AdS_5 \times S^5$ background, it is tempting to try to identify this non-linear topological A-model at large but finite radius with the $AdS_5 \times S^5$ sigma model at small but non-zero $T_{\alpha\beta}{}^c$. However, this identification does not seem possible since when $T_{\alpha\beta}{}^c$ is non-zero, the $AdS_5 \times S^5$ sigma model contains a Wess-Zumino-Witten term which is antisymmetric under exchange of z and $\overline{z}$ and which breaks $\mathrm{SU}(2,2|4)$ down to $PSU(2,2|4)$. On the other hand, the non-linear topological A-model is symmetric under exchange of z and $\overline{z}$ and preserves $\mathrm{SU}(2,2|4)$ invariance. So it appears that the $AdS_5 \times S^5$ sigma model and the non-linear topological A-model can only be identified in the limit where $T_{\alpha\beta}{}^c = 0$ in the $AdS_5 \times S^5$ model and where the radius is infinite in the non-linear model.

4.1 Superspace action

Although the non-linear topological A-model has both N=(2,2) worldsheet supersymmetry and $\mathrm{SU}(2,2|4)$ invariance, both these symmetries can not be simultaneously made manifest. The worldsheet supersymmetry can be made manifest by expressing the non-linear action in superspace as

$$\begin{aligned} S &= \frac{1}{\Lambda}\int d^2z d^4\kappa \Big[\overline{\Theta}_{rj}\Theta^{jr} - \frac{1}{2R^2}\overline{\Theta}_{rj}\Theta^{js}\overline{\Theta}_{sk}\Theta^{kr} + \frac{1}{3R^4}\overline{\Theta}_{rj}\Theta^{js}\overline{\Theta}_{sk}\Theta^{kt}\overline{\Theta}_{tl}\Theta^{lr} + \cdots\Big] \qquad (4.1)\\ &= \frac{R^2}{\Lambda}\int d^2z d^4\kappa \; Tr\left[\log\left(1 + \frac{1}{R^2}\overline{\Theta}\Theta\right)\right] \end{aligned}$$

where Θ_{rj} and $\overline{\Theta}_{jr}$ are the same superfields as in (3.23), and R is the radius of this model which is unrelated to the $AdS_5 \times S^5$ radius r. In the limit $R \to \infty$, this non-linear model reduces to the linear topological A-model of (3.24). The non-linear action of (4.1) is invariant under the same $\mathrm{SU}(2,2) \times \mathrm{SU}(4) \times \mathrm{U}(1)$ transformations as (3.25), but the fermionic isometries of (3.26) are modified to

$$\delta\Theta^{rj} = \alpha^{rj} + \frac{1}{R^2}\Theta^{rk}\overline{\alpha}_{ks}\Theta^{sj}, \quad \delta\overline{\Theta}_{jr} = \overline{\alpha}_{jr} + \frac{1}{R^2}\overline{\Theta}_{js}\alpha^{sk}\overline{\Theta}_{kr}, \qquad (4.2)$$

which close with the bosonic isometries into the $\mathrm{SU}(2,2|4)$ supergroup.

4.2 Coset action

These $\mathrm{SU}(2,2|4)$ isometries can be made manifest by rescaling $\Theta^{rj} \to R\Theta^{rj}$ and $\overline{\Theta}_{jr} \to R\overline{\Theta}_{jr}$ and writing the non-linear action in terms of the component fields $(\theta^{rj}, \overline{\theta}_{jr}, Z^{rj}, \overline{Z}_{jr}, Y_{jr}, \overline{Y}^{rj})$ using a coset space construction. The coset G will be defined to take values in $\frac{PSU(2,2|4)}{\mathrm{SU}(2,2)\times\mathrm{SU}(4)}$, and since the coset has only fermionic elements, G can be gauged to the form

$$G_j^k = \delta_j^k, \quad G_s^r = \delta_s^r, \quad G^{rj} = \theta^{rj}, \quad G_{jr} = \overline{\theta}_{jr}. \qquad (4.3)$$

In terms of the left-invariant currents $J^A = (G^{-1}\partial G)^A$ and $\overline{J}^A = (G^{-1}\overline{\partial} G)^A$ where A is an $\mathrm{SU}(2,2|4)$ index, the action is

$$S = \frac{R^2}{\Lambda}\int d^2z \Big[(\overline{J}-\overline{\mathcal{A}})_s^r (J-\mathcal{A})_r^s - (\overline{J}-\overline{\mathcal{A}})_j^k (J-\mathcal{A})_k^j \tag{4.4}$$
$$+\overline{J}_{jr}J^{rj} + Y_{jr}(\overline{\partial}Z + \overline{\mathcal{A}}Z)^{rj} + \overline{Y}^{rj}(\partial\overline{Z} - \mathcal{A}\overline{Z})_{jr}\Big]$$
$$= \frac{R^2}{\Lambda}\int d^2z \Big[\overline{J}_{jr}J^{rj} + Y_{jr}\overline{\nabla}Z^{rj} + \overline{Y}^{rj}\nabla\overline{Z}_{jr} + Y_{jr}Z^{rk}\overline{Z}_{ks}\overline{Y}^{sj} - Z^{rj}Y_{js}\overline{Y}^{sk}\overline{Z}_{kr}\Big] \tag{4.5}$$

where $(\mathcal{A}^A, \overline{\mathcal{A}}^A)$ are $\mathrm{SU}(2,2)\times\mathrm{SU}(4)$ gauge fields, $\overline{\nabla}Z^{jr} = \overline{\partial}Z^{jr} + \overline{J}_s^r Z^{js} + \overline{J}_k^j Z^{kr}$, and $\nabla\overline{Z}_{rj} = \partial\overline{Z}_{rj} - J_r^s\overline{Z}_{sj} - J_j^k\overline{Z}_{rk}$. Note that

$$\overline{J}_{jr}J^{rj} - J_{jr}\overline{J}^{rj} = \partial\overline{J}_{\mathrm{U}(1)} - \overline{\partial}J_{\mathrm{U}(1)} \tag{4.6}$$

is a total derivative where $J_{\mathrm{U}(1)}$ is the "bonus" U(1) current, so the term $\int d^2z\overline{J}_{jr}J^{rj}$ is symmetric under exchange of z and $\overline{z}$.

Although $\mathrm{SU}(2,2|4)$ invariance is manifest in the action of (4.4), N=(2,2) worldsheet supersymmetry is not manifest. Nevertheless, one can easily construct the twisted N=(2,2) worldsheet supersymmetry generators as

$$Q = \int dz Z^{rj}J_{jr}, \quad \overline{Q} = \int d\overline{z}\overline{Z}_{jr}\overline{J}^{rj}, \quad b = Y_{jr}J^{rj}, \quad \overline{b} = \overline{Y}^{rj}\overline{J}_{jr}. \tag{4.7}$$

After parameterizing G as in (4.3), the action of (4.5) coincides with the superspace action of (4.1) after integrating out the auxiliary fields f^{rj} and $\overline{f}_{jr}$.

4.3 One-loop conformal invariance

To show that the non-linear topological A-model has no one-loop conformal anomaly, one can either use the superspace version of the action of (4.1) and compute $\log\ \det(\partial\overline{\partial}K)$ where K is the Kahler potential, or one can use the coset version of the action of (4.5) and compute the anomaly with the background field method of [20] and [4]. Absence of this anomaly is necessary for the topological twisting to be consistent at the quantum level.

Using the superspace action of (4.1), $K = Tr\ \log(1+\overline{\Theta}\Theta)$ implies that

$$\partial_{ks}\overline{\partial}^{rj}K = \partial_{ks}[\Theta^{rl}[(1+\overline{\Theta}\Theta)^{-1}]_l^j] \tag{4.8}$$
$$= \delta_s^r[(1+\overline{\Theta}\Theta)^{-1}]_k^j - \Theta^{rl}[(1+\overline{\Theta}\Theta)^{-1}]_l^m\overline{\Theta}_{ms}[(1+\overline{\Theta}\Theta)^{-1}]_k^j$$
$$= [(1+\Theta\overline{\Theta})^{-1}]_s^r[(1+\overline{\Theta}\Theta)^{-1}]_k^j.$$

So there is no conformal anomaly since

$$\log\ \det(\partial_{ks}\overline{\partial}^{rj}K) = \log\ \det[(1+\Theta\overline{\Theta})^{-1}] + \log\ \det[(1+\overline{\Theta}\Theta)^{-1}] \tag{4.9}$$
$$= -Tr\ \log(1+\Theta\overline{\Theta}) - Tr\ \log(1+\overline{\Theta}\Theta)$$
$$= -Tr\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(\Theta\overline{\Theta})^n + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(\overline{\Theta}\Theta)^n\right] = 0$$

where we have used that $Tr[(\Theta\overline{\Theta})^n] = -Tr[(\overline{\Theta}\Theta)^n]$ for $n > 0$.

Using the background field method for the coset action of (4.5), the matter sector of $\int d^2z \overline{J}_{jr} J^{rj}$ contributes no conformal anomaly since, when G/H is a symmetric space, the G/H coset model has the same conformal anomaly as the principal chiral model based on G [20]. In this case, $PSU(2,2|4)/(\mathrm{SU}(2,2)\times\mathrm{SU}(4))$ is a symmetric space, and the principal chiral model based on $PSU(2,2|4)$ has no conformal anomaly [21].

Furthermore, the ghost sector of (4.5) contributes no conformal anomaly because of a cancellation between the $Y_{jr}\overline{\nabla}Z^{rj} + \overline{Y}^{rj}\nabla\overline{Z}_{jr}$ contribution and the $Y_{jr}Z^{rk}\overline{Z}_{ks}\overline{Y}^{sj} - Z^{rj}Y_{js}\overline{Y}^{sk}\overline{Z}_{kr}$ contribution. As shown in [4], the $Y_{jr}\overline{\nabla}Z^{rj} + \overline{Y}^{rj}\nabla\overline{Z}_{jr}$ term contributes an anomaly proportional to the dual coxeter number of the group, and $Y_{jr}Z^{rk}\overline{Z}_{ks}\overline{Y}^{sj} - Z^{rj}Y_{js}\overline{Y}^{sk}\overline{Z}_{kr}$ contributes an anomaly proportional to the level k in the OPE of the Lorentz currents. In the $AdS_5 \times S^5$ case, the relevant group was $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ with dual coxeter number 3, which cancels the level $k = -3$ in the OPE of the Lorentz currents constructed from pure spinors [4]. In this case, the relevant group is $\mathrm{SU}(2,2)\times\mathrm{SU}(4)$ with dual coxeter number 4, which cancels the level $k = -4$ in the OPE of Lorentz currents constructed from unconstrained bosonic spinors.

4.4 Open string sector

Just as d=3 Chern-Simons theory is described by the open string sector of a topological A-model [13], it will be shown that the open string sector of the non-linear topological A-model of (4.1) describes N=4 d=4 super-Yang-Mills. The open string boundary condition for the A-model of (4.1) will be defined as

$$\overline{\Theta}_{jr} = \delta_{jk}\epsilon_{rs}\Theta^{sk} \tag{4.10}$$

where ϵ_{rs} is an antisymmetric tensor which breaks $\mathrm{SU}(2,2)$ to $\mathrm{SO}(3,2)$ and δ_{jk} is a symmetric tensor which breaks $\mathrm{SU}(4)$ to $\mathrm{SO}(4)$. The boundary condition of (4.10) is similar to the open string boundary condition for the Chern-Simons topological string which is $\overline{X}_I = \delta_{IJ}X^J$ for $I,J = 1$ to 3. Note that the open string boundary for the A-model is defined by

$$z = \overline{z}, \quad \kappa_+ = \overline{\kappa}_-, \quad \overline{\kappa}_+ = \kappa_-, \tag{4.11}$$

so (4.10) implies that

$$\overline{\theta}_{jr} = \delta_{jk}\epsilon_{rs}\theta^{sk}, \quad \overline{Z}_{jr} = \delta_{jk}\epsilon_{rs}Z^{sk}, \quad Y_{jr} = \delta_{jk}\epsilon_{rs}\overline{Y}^{sk}. \tag{4.12}$$

The boundary condition of (4.10) breaks half of the fermionic isometries and reduces the $\mathrm{SU}(2,2|4)$ supergroup of isometries to the supergroup $\mathrm{OSp}(4|4)$. This supergroup contains $\mathrm{SO}(3,2)\times\mathrm{SO}(4)$ bosonic isometries and 16 fermionic isometries, and is the N=4 supersymmetry algebra on AdS_4.

To show that the BRST cohomology of open string states in this model describes N=4 d=4 super-Yang-Mills, it will be assumed that, as in the topological A-model for Chern-Simons, the cohomology in the closed string sector is trivial. This assumption is reasonable since N=(2,2) worldsheet supersymmetric D-terms are BRST-trivial, and there are naively no global obstructions to writing supersymmetric expressions involving fermionic superfields as superspace D-terms. However, since the A-model of (4.1) is constructed from fermionic superfields in a non-conventional manner, there might be unexpected subtleties in the model which invalidate this assumption.

With this assumption, the cohomology computation in the open string sector is independent of Λ and R in (4.1), and can be performed at $\Lambda = 0$ where only the constant modes of Θ^{rj} contribute. Furthermore, if the closed string sector has no cohomology, the open string physical states should be independent of $\mathrm{SU}(2,2|4)/\mathrm{OSp}(4|4)$ rotations which modify the D-brane boundary conditions of (4.10). So although only OSp(4|4) symmetry is manifest in the open topological A-model, the physical spectrum should be invariant under the full $\mathrm{SU}(2,2|4)$ supergroup.

After imposing the open string boundary condition of (4.10) and restricting to constant worldsheet modes, the superspace action of (4.1) reduces to

$$S = R^2 \int d\tau d^2\kappa \; Tr[D_+\Theta(1+\Theta\Theta)^{-1}D_-\Theta(1+\Theta\Theta)^{-1}] \tag{4.13}$$

where $\Theta_{jr} = \delta_{jk}\epsilon_{rs}\Theta^{sk}$ is an N=2 superfield whose component expansion is

$$\Theta^{rj} = \theta^{jr} + \kappa_+ Y^{rj} + \kappa_- Z^{rj} + \kappa_+\kappa_- f^{rj}, \tag{4.14}$$

and $D_\pm = \frac{\partial}{\kappa^\pm} + \kappa^\mp \frac{\partial}{\partial\tau}$. Alternatively, using the coset construction, the action of (4.5) reduces to

$$\begin{aligned} S &= R^2 \int d\tau \left[\epsilon_{rs} J^{rj} J^{sj} + (J-\mathcal{A})^r_s (J-\mathcal{A})^s_r - (J-\mathcal{A})^k_j (J-\mathcal{A})^j_k + Y_{jr}\left(\frac{\partial}{\partial\tau} Z + \mathcal{A}Z\right)^{rj}\right] \\ &= R^2 \int d\tau [\epsilon_{rs} J^{rj} J^{sj} + Y_{jr}(\nabla Z)^{rj} + (YZ)^k_j (YZ)^j_k - (YZ)^s_r (YZ)^r_s], \end{aligned} \tag{4.15}$$

where $J^A = (G^{-1}\frac{\partial}{\partial\tau}G)^A$ are left-invariant currents taking values in the Lie algebra of OSp(4|4), $G(\theta)$ takes values in the coset $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,2)\times\mathrm{SO}(4)}$, $A = ([rs],[jk],jr)$ labels the OSp(4|4) generators, $r = 1$ to 4 labels $Sp(4)$ indices which are raised and lowered using the antisymmetric metric ϵ^{rs}, $j = 1$ to 4 labels SO(4) indices which are raised and lowered using δ_{jk}, $\mathcal{A}^A$ is an $Sp(4)\times\mathrm{SO}(4)$ worldline gauge field, and $(\nabla Z)^{rj} = \frac{\partial}{\partial\tau}Z^{rj} + J^r_s Z^{sj} + J^j_k Z^{rk}$. The N=2 worldline supersymmetry generators for this action are

$$Q = Z^{rj} J_{jr}, \quad b = Y_{jr} J^{rj}. \tag{4.16}$$

5. Cohomology of open topological A-model

Before showing that the BRST cohomology of the worldline action of (4.15) describes N=4 d=4 super-Yang-Mills, it will be useful to review the superspace description of on-shell super-Yang-Mills.

5.1 On-shell super-Yang-Mills in superspace

In ten flat dimensions, on-shell super-Yang-Mills is described by a spinor superfield $A_\alpha(x,\theta)$ where $\alpha = 1$ to 16. This superfield can be understood as a spinor connection which covariantizes the superspace derivative $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \gamma^c_{\alpha\beta}\frac{\partial}{\partial x^c}$ to $\nabla_\alpha = D_\alpha - A_\alpha(x,\theta)$. Since $\{D_\alpha, D_\beta\} = \gamma^c_{\alpha\beta}\frac{\partial}{\partial x^c}$, it is natural to impose that A_α is defined such that [22]

$$\{\nabla_\alpha, \nabla_\beta\} = \gamma^c_{\alpha\beta}\nabla_c \tag{5.1}$$

where $\nabla_c = \frac{\partial}{\partial x^c} - A_c(x,\theta)$ and $A_c(x,\theta)$ is a vector connection whose $\theta = 0$ component is the usual gauge field.

These spinor and vector superspace connections are defined up to the gauge transformation

$$\delta A_\alpha = \nabla_\alpha\Omega, \quad \delta A_c = \nabla_c\Omega \tag{5.2}$$

where Ω is a scalar superfield, and the Bianchi identity of (5.1) implies that

$$D_\alpha A_\beta + D_\beta A_\alpha - \{A_\alpha, A_\beta\} = \gamma^c_{\alpha\beta}A_c. \tag{5.3}$$

Equation (5.3) implies that A_c is determined from A_α and that A_α must satisfy the constraint

$$(\gamma^{abcde})^{\alpha\beta}\left(D_\alpha A_\beta - \frac{1}{2}\{A_\alpha, A_\beta\}\right) = 0 \tag{5.4}$$

for any five-form direction $abcde$ [23].

The constraint of (5.4) together with the gauge invariance of (5.2) implies that $A_\alpha(x,\theta)$ can be gauged to the form

$$A_\alpha(x,\theta) = a_c(x)(\gamma^c\theta)_\alpha + \xi^\beta(x)(\gamma^c\theta)_\beta(\gamma_c\theta)_\alpha + \cdots \tag{5.5}$$

where $a_c(x)$ and $\xi^\alpha(x)$ are the on-shell gluon and gluino, and ... involves spacetime derivatives of $a_c(x)$ and $\xi^\alpha(x)$.

To describe N=4 d=4 super-Yang-Mills, one simply decomposes the d=10 vectors and spinors into d=4 vectors, scalars and spinors in the usual manner as

$$\theta^\alpha \to (\theta^{\mu j}, \overline{\theta}^{\dot\mu}_j), \quad A_\alpha \to (A_{\mu j}, \overline{A}^j_{\dot\mu}), \quad A_c \to (A_m, A_{[jk]}) \tag{5.6}$$

where $m = 0$ to 3, $\mu, \dot\mu = 1$ to 2, $j = 1$ to 4, and $[jk] = 1$ to 6. The corresponding covariant spinor and vector derivatives satisfy the Bianchi identities

$$\{\nabla_{\mu j}, \overline{\nabla}^k_{\dot\mu}\} = \delta^k_j\sigma^m_{\mu\dot\mu}\nabla_m, \quad \{\nabla_{\mu j}, \nabla_{\nu k}\} = \epsilon_{\mu\nu}A_{[jk]}, \quad \{\overline{\nabla}^j_{\dot\mu}, \overline{\nabla}^k_{\dot\nu}\} = \frac{1}{2}\epsilon_{\dot\mu\dot\nu}\epsilon^{hijk}A_{[hi]}, \tag{5.7}$$

where $\sigma^m_{\mu\dot\mu}$ are the d=4 Pauli matrices. So the N=4 d=4 spinor connections satisfy the equations

$$D_{\mu j}\overline{A}^k_{\dot\nu} + \overline{D}^k_{\dot\nu}A_{\mu j} - \{A_{\mu j}, \overline{A}^k_{\dot\nu}\} = \delta^k_j\sigma^m_{\mu\dot\nu}A_m, \tag{5.8}$$
$$D_{(\mu j}A_{\nu k)} - \{A_{\mu j}, A_{\nu k}\} = \epsilon_{\mu\nu}A_{[jk]}, \quad \overline{D}^{(\dot\mu j}\overline{A}^{\dot\nu k)} - \{\overline{A}^{\dot\mu j}, \overline{A}^{\dot\nu k}\} = \frac{1}{2}\epsilon^{\dot\mu\dot\nu}\epsilon^{hijk}A_{[hi]},$$

JHEP08(2007)011

and the gauge transformations

$$\delta A_{\mu j}=\nabla_{\mu j}\Omega,\quad \delta\overline{A}^{j}_{\dot\mu}=\overline{\nabla}^{j}_{\dot\mu}\Omega,\quad \delta A_m=\nabla_m\Omega. \tag{5.9}$$

Since N=4 d=4 super-Yang-Mills is superconformally invariant, the Bianchi identities of (5.7) are valid both in flat d=4 Minkowski space and in AdS_4 space. The only difference is that in a flat background, the superspace derivatives are

$$D_{\mu j}=\frac{\partial}{\partial\theta^{\mu j}}+\overline{\theta}^{\dot\mu}_{j}\sigma^{m}_{\mu\dot\mu}\frac{\partial}{\partial x^m},\quad \overline{D}^{j}_{\dot\mu}=\frac{\partial}{\partial\overline{\theta}^{\dot\mu}_{j}}+\theta^{\mu j}\sigma^{m}_{\mu\dot\mu}\frac{\partial}{\partial x^m},\quad D_m=\frac{\partial}{\partial x^m}, \tag{5.10}$$

whereas in an AdS_4 background,

$$D_A=E^M_A\frac{\partial}{\partial Y^M}+w^{[mn]}_A M_{[mn]}+w^{[jk]}_A M_{[jk]} \tag{5.11}$$

where E^M_A is the AdS_4 super-vierbein, $Y^M=(y^m,\xi^{\mu j},\overline{\xi}^{\dot\mu}_j)$ are the AdS_4 superspace coordinates, w_A is the AdS_4 super-connection, and $M_{[mn]}$ and $M_{[jk]}$ are the SO(3,1) and SO(4) generators. As will be shown in subsection 5.3, the AdS_4 super-vierbein and super-connection can be naturally constructed from a supercoset $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(4)}$ in the same manner as the $AdS_5\times S^5$ super-vierbein and super-connection are constructed from the $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ supercoset.

5.2 First-quantized description of $N=4$ $d=4$ super-Yang-Mills

Just as d=3 Chern-Simons can be obtained by quantizing the worldline action $\int d\tau(\frac{1}{2}\frac{\partial x^I}{\partial\tau}\frac{\partial x_I}{\partial\tau}+\overline{\psi}_I\frac{\partial}{\partial\tau}\psi^I)$ with the BRST operator $Q=\psi^I\frac{\partial}{\partial\tau}x_I$ where $I=1$ to 3, d=10 super-Yang-Mills can be obtained by quantizing the worldline action $\int d\tau(\frac{1}{2}\frac{\partial x^c}{\partial\tau}\frac{\partial x_c}{\partial\tau}+p_\alpha\frac{\partial}{\partial\tau}\theta^\alpha+w_\alpha\frac{\partial}{\partial\tau}\lambda^\alpha)$ with the BRST operator $Q=\lambda^\alpha d_\alpha$ where $d_\alpha=p_\alpha+(\gamma_c\theta)_\alpha\frac{\partial}{\partial\tau}x^c$ and λ^α is a pure spinor satisfying $\lambda\gamma^c\lambda=0$ for $c=0$ to 9 [15, 23].

At ghost-number one, the states in the cohomology of $Q=\lambda^\alpha d_\alpha$ are described by $V=\lambda^\alpha A_\alpha(x,\theta)$ where $A_\alpha(x,\theta)$ is a spinor superfield. $QV=0$ implies that $\lambda^\alpha\lambda^\beta D_\beta A_\alpha=0$ where $D_\alpha=\frac{\partial}{\partial\theta^\alpha}+(\gamma^c\theta)\frac{\partial}{\partial x^c}$, and since $\lambda\gamma^c\lambda=0$, $\lambda^\alpha\lambda^\beta D_\beta A_\alpha=0$ implies that $D_\alpha A_\beta+D_\beta A_\alpha=\gamma^c_{\alpha\beta}A_c$ for some A_c. Also, $\delta V=Q\Omega$ implies that $\delta A_\alpha=D_\alpha\Omega$. By comparing with (5.3) and (5.2), one sees that $A_\alpha(x,\theta)$ describes the linearized on-shell d=10 super-Yang-Mills fields.

The structure of $V=\lambda^\alpha A_\alpha(x,\theta)$ in d=10 super-Yang-Mills using the BRST operator $Q=\lambda^\alpha d_\alpha$ closely resembles the structure of $V=\psi^I A_I(x)$ in Chern-Simons theory using the BRST operator $Q=\psi^I\frac{\partial}{\partial\tau}x_I$. In Chern-Simons theory, $QV=0$ implies that $\partial_I A_J-\partial_J A_I=0$ and $\delta V=Q\Omega$ implies that $\delta A_I=\partial_I\Omega$. Furthermore, as in Chern-Simons theory, the super-Yang-Mills ghost is described by the BRST cohomology at ghost-number zero, the super-Yang-Mills fields are described by the BRST cohomology at ghost-number one, the super-Yang-Mills antifields are described by the BRST cohomology at ghost-number two, and the super-Yang-Mills antighost is described by the BRST cohomology at ghost-number three [15]. This structure can be seen from the Batalin-Vilkovisky action for d=10 super-Yang-Mills which can be written in the Chern-Simons-like form $S=\langle VQV+\frac{2}{3}V^3\rangle$ using the normalization convention that $\langle(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta)(\theta\gamma_{abc}\theta)\rangle=1$.

This construction for d=10 super-Yang-Mills is easily generalized to N=4 d=4 super-Yang-Mills by eliminating six of the ten x's and decomposing the d=10 spinors into N=4 d=4 spinors as

$$\theta^\alpha \to (\theta^{\mu j}, \overline{\theta}^{\dot\mu}_j), \quad p_\alpha \to (p_{\mu j}, \overline{p}^j_{\dot\mu}), \quad \lambda^\alpha \to (\lambda^{\mu j}, \overline{\lambda}^{\dot\mu}_j), \quad w_\alpha \to (w_{\mu j}, \overline{w}^j_{\dot\mu}), \tag{5.12}$$

where $\mu, \dot\mu = 1$ to 2 and $j = 1$ to 4. The pure spinor condition $\lambda\gamma^c\lambda = 0$ implies that $\lambda^{\mu j}$ and $\overline{\lambda}^{\dot\mu}_j$ satisfy the constraints

$$\lambda^{\mu j}\overline{\lambda}^{\dot\mu}_j = 0, \tag{5.13}$$

$$\epsilon_{\mu\nu}\lambda^{\mu j}\lambda^{\nu k} = \frac{1}{2}\epsilon_{\dot\mu\dot\nu}\epsilon^{hijk}\overline{\lambda}^{\dot\mu}_h\overline{\lambda}^{\dot\nu}_i. \tag{5.14}$$

Although (5.13) and (5.14) contain ten constraints, only five of these constraints are independent. This is easy to verify since $\lambda^{\mu j}\overline{\lambda}^{\dot\mu}_j = 0$ implies that $\overline{\lambda}^{\dot\rho}_j(\epsilon_{\mu\nu}\lambda^{\mu j}\lambda^{\nu k}) = 0$, which implies that

$$\epsilon_{\mu\nu}\lambda^{\mu j}\lambda^{\nu k} = \frac{1}{2}e^{2\phi}\epsilon^{hijk}\epsilon_{\dot\mu\dot\nu}\overline{\lambda}^{\dot\mu}_h\overline{\lambda}^{\dot\nu}_i \tag{5.15}$$

for some ϕ. So if the four constraints in (5.13) are satisfied, any one of the constraints in (5.14) imply that $\phi = 0$, which implies that the remaining five constraints in (5.14) are satisfied.

Since the four constraints of (5.13) are almost strong enough to define an N=4 d=4 pure spinor, it will be convenient to define a "semi-pure" spinor $(\lambda'^{\mu j}, \overline{\lambda}'^{\dot\mu}_j)$ which is only required to satisfy the four constraints of (5.13) that

$$\lambda'^{\mu j}\overline{\lambda}'^{\dot\mu}_j = 0. \tag{5.16}$$

A semi-pure spinor has 12 independent components and is related to a pure spinor $(\lambda^{\mu j}, \overline{\lambda}^{\dot\mu}_j)$ by a U(1) "R-transformation" as

$$\lambda'^{\mu j} = e^{\frac{\phi}{2}}\lambda^{\mu j}, \quad \overline{\lambda}'^{\dot\mu}_j = e^{-\frac{\phi}{2}}\overline{\lambda}^{\dot\mu}_j \tag{5.17}$$

where ϕ is determined from

$$e^{2\phi} = \frac{\epsilon_{\mu\nu}\lambda'^{\mu j}\lambda'^{\nu k}}{\frac{1}{2}\epsilon^{hijk}\epsilon_{\dot\mu\dot\nu}\overline{\lambda}'^{\dot\mu}_h\overline{\lambda}'^{\dot\nu}_i}. \tag{5.18}$$

In flat d=4 Minkowski space, the worldline action for N=4 d=4 super-Yang-Mills will be defined as

$$S = \int d\tau\left(\frac{1}{2}\frac{\partial x^m}{\partial\tau}\frac{\partial x_m}{\partial\tau} + p_{\mu j}\frac{\partial}{\partial\tau}\theta^{\mu j} + \overline{p}^j_{\dot\mu}\frac{\partial}{\partial\tau}\overline{\theta}^{\dot\mu}_j + w'_{\mu j}\frac{\partial}{\partial\tau}\lambda'^{\mu j} + \overline{w}'^j_{\dot\mu}\frac{\partial}{\partial\tau}\overline{\lambda}'^{\dot\mu}_j\right] \tag{5.19}$$

with the BRST operator

$$Q = \lambda'^{\mu j}d_{\mu j} + \overline{\lambda}'^{\dot\mu}_j\overline{d}^j_{\dot\mu} \tag{5.20}$$

where $d_{\mu j} = p_{\mu j} + \sigma^m_{\mu\dot\mu}\overline{\theta}^{\dot\mu}_j\frac{\partial x_m}{\partial\tau}$, $\overline{d}^j_{\dot\mu} = \overline{p}^j_{\dot\mu} + \sigma^m_{\mu\dot\mu}\theta^{\mu j}\frac{\partial x_m}{\partial\tau}$, and $\lambda'^{\mu j}$ and $\overline{\lambda}'^{\dot\mu}_j$ are semi-pure spinors satisfying (5.16). Note that $Q^2 = 0$ since $\{d_{\mu j}, \overline{d}^k_{\dot\mu}\} = \delta^k_j\sigma^m_{\mu\dot\mu}\frac{\partial x_m}{\partial\tau}$, and that $w'_{\mu j}$ and $\overline{w}'^j_{\dot\mu}$ can only appear in combinations which are invariant under the gauge transformations

$$\delta w'_{\mu j} = \xi_m\sigma^m_{\mu\dot\mu}\overline{\lambda}'^{\dot\mu}_j, \quad \delta\overline{w}'^j_{\dot\mu} = \xi_m\sigma^m_{\mu\dot\mu}\lambda'^{\mu j}. \tag{5.21}$$

JHEP08(2007)011

The action and BRST operator of (5.19) and (5.20) are invariant under the U(1) R-transformation

$$\theta^{\mu j} \to c\theta^{\mu j}, \qquad \overline{\theta}^{\dot\mu}_j \to c^{-1}\overline{\theta}^{\dot\mu}_j, \qquad p_{\mu j} \to c^{-1}p_{\mu j}, \qquad \overline{p}^j_{\dot\mu} \to c\overline{p}^j_{\dot\mu}, \tag{5.22}$$
$$\lambda'^{\mu j} \to c\lambda'^{\mu j}, \qquad \overline{\lambda}'^{\dot\mu}_j \to c^{-1}\overline{\lambda}'^{\dot\mu}_j, \qquad w'_{\mu j} \to c^{-1}w'_{\mu j}, \qquad \overline{w}'^j_{\dot\mu} \to c\overline{w}'^j_{\dot\mu},$$

however, N=4 d=4 super-Yang-Mills does not contain such a U(1) symmetry. Since the variable ϕ of (5.18) transforms under (5.22) as

$$\phi \to \phi + \frac{1}{2}\log\ c, \tag{5.23}$$

ϕ can be interpreted as a "compensator" for U(1) R-transformations which cancels the U(1) R-transformation of $\theta^{\mu j}$ and $\overline{\theta}^{\dot\mu}_j$. Physical states will therefore be defined as states of +1 ghost-number in the BRST cohomology which are invariant under the R-transformation of (5.22).

At ghost-number one, R-invariant states are described by

$$V = e^{-\frac{\phi}{2}}\lambda'^{\mu j}A_{\mu j}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta}e^{\frac{\phi}{2}}) + e^{\frac{\phi}{2}}\overline{\lambda}'^{\dot\mu}_j\overline{A}^j_{\dot\mu}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta}e^{\frac{\phi}{2}}) \tag{5.24}$$

where ϕ is defined in (5.18) and cancels the R-transformation of λ' and θ. In other words,

$$V = \lambda'^{\mu j}A'_{\mu j}(x, \theta', \overline{\theta}') + \overline{\lambda}'^{\dot\mu}_j\overline{A}'^j_{\dot\mu}(x, \theta', \overline{\theta}') \tag{5.25}$$

where $A'_{\mu j}(x, \theta', \overline{\theta}') = e^{-\frac{\phi}{2}}A_{\mu j}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta}e^{\frac{\phi}{2}})$ and $\overline{A}'^j_{\dot\mu}(x, \theta', \overline{\theta}') = e^{\frac{\phi}{2}}\overline{A}^j_{\dot\mu}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta}e^{\frac{\phi}{2}})$ are the R-transformed versions of $A_{\mu j}(x, \theta, \overline{\theta})$ and $\overline{A}^j_{\dot\mu}(x, \theta, \overline{\theta})$ using the R-parameter $c = e^{-\frac{\phi}{2}}$ in (5.22). The equation $QV = 0$ implies that

$$e^{-\phi}\lambda'^{\mu j}\lambda'^{\nu k}D_{\mu j}A_{\nu k} + e^{\phi}\overline{\lambda}'^{\dot\mu}_j\overline{\lambda}'^{\dot\nu}_k\overline{D}^k_{\dot\mu}\overline{A}^k_{\dot\nu} + \lambda'^{\mu j}\overline{\lambda}'^{\dot\nu}_k(D_{\mu j}\overline{A}^k_{\dot\nu} + \overline{D}^k_{\dot\nu}A_{\mu j}) = 0, \tag{5.26}$$

which implies using the pure spinor constraints of (5.13) - (5.18) that

$$D_{\mu j}\overline{A}^k_{\dot\nu} + \overline{D}^k_{\dot\nu}A_{\mu j} = \delta^k_j\sigma^m_{\mu\dot\nu}A_m, \quad D_{(\mu j}A_{\nu k)} = \epsilon_{\mu\nu}A_{[jk]}, \quad \overline{D}^{(\dot\mu j}\overline{A}^{\dot\nu k)} = \frac{1}{2}\epsilon^{\dot\mu\dot\nu}\epsilon^{hijk}A_{[hi]}, \tag{5.27}$$

for some superfields $A_m(x, \theta, \overline{\theta})$ and $A_{[jk]}(x, \theta, \overline{\theta})$. Furthermore, the gauge transformation $\delta V = Q\Omega(x, e^{-\frac{\phi}{2}}\theta, e^{\frac{\phi}{2}}\overline{\theta})$ implies that

$$\delta A_{\mu j} = D_{\mu j}\Omega, \quad \delta\overline{A}^j_{\dot\mu} = \overline{D}^j_{\dot\mu}\Omega, \quad \delta A_m = \partial_m\Omega. \tag{5.28}$$

So when V of (5.24) is in the BRST cohomology, $A_{\mu j}$ and $\overline{A}^j_{\dot\mu}$ satisfy the linearized N=4 d=4 super-Yang-Mills equations and gauge invariances of (5.8) and (5.9) in flat Minkowski space.

JHEP08(2007)011

5.3 $N=4$ $d=4$ super-Yang-Mills in AdS_4

To generalize this construction to N=4 d=4 super-Yang-Mills in an AdS_4 background, one needs to modify the worldline action and BRST operator of (5.19) and (5.20) to be OSp(4|4) invariant. This can be done using a coset construction based on $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(4)}$ which contains four bosonic generators and sixteen fermionic generators. As in the $AdS_5\times S^5$ construction, it is convenient to define left-invariant currents $J^A=(g^{-1}\frac{\partial}{\partial\tau}g)^A$ where $g(x,\theta)$ takes values in the $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(4)}$ coset, $A=(m,[mn],[jk],rj)$ label the OSp(4,4) generators, $m=0$ to 3 label the "translation" generators, $[mn]$ and $[jk]$ label the SO(3,1) and SO(4) generators, and rj label the "supersymmetry" generators for $r=1$ to 4 and $j=1$ to 4. Note that the two-component μ index corresponds to $r=1,2$, the two-component $\dot\mu$ index corresponds to $r=3,4$, and the antisymmetric ϵ_{rs} tensor has non-zero components $\epsilon_{12}=-\epsilon_{21}=\epsilon_{34}=-\epsilon_{43}=1$.

The OSp(4|4)-invariant worldline action is

$$S=R^2\int d\tau\left[\frac{1}{4}J^mJ_m+\epsilon_{rs}J^{rj}J^{sj}+w'_{rj}\left(\frac{\partial}{\partial\tau}\lambda'+\mathcal{A}\lambda'\right)^{rj}\right. \tag{5.29}$$
$$\left.+(J^{[mn]}-\mathcal{A}^{[mn]})(J_{[mn]}-\mathcal{A}_{[mn]})-(J^{[jk]}-\mathcal{A}^{[jk]})(J_{[jk]}-\mathcal{A}_{[jk]})\right]$$
$$=R^2\int d\tau\left[\frac{1}{4}J^mJ_m+\epsilon_{rs}J^{rj}J^{sj}+w'_{rj}(\nabla\lambda')^{rj}+(w'\lambda')_j^k(w'\lambda')_k^j-(w'\sigma^{mn}\lambda')(w'\sigma_{mn}\lambda')\right],$$

where $(w'\lambda')_j^k=w'_{rj}\lambda'^{rk}$, $(w'\sigma^{mn}\lambda')=(\sigma^{mn})_s^r w'_{rj}\lambda'^{sj}$ and $(\nabla\lambda')^{rj}=\frac{\partial}{\partial\tau}\lambda'^{rj}+\frac{1}{2}J_{[mn]}(\sigma^{[mn]})_s^r\lambda'^{sj}+J_k^j\lambda'^{rk}$. This action is invariant under local $\mathrm{SO}(3,1)\times\mathrm{SO}(4)$ transformations where λ' and w' transform covariantly, and is also invariant under the BRST transformations

$$\delta g=g(\epsilon\lambda'^{rj}T_{rj}),\quad \delta w'_{rj}=\epsilon J_{rj}, \tag{5.30}$$

generated by the BRST operator $Q=\lambda'^{rj}J_{rj}$ where T_{rj} are the fermionic generators of OSp(4|4).

Defining the ghost-number one vertex operator as

$$V=\lambda'^{rj}A'_{rj}=\lambda'^{\mu j}A'_{\mu j}+\overline{\lambda}'^{\dot\mu}_{j}\overline{A}'^{j}_{\dot\mu}, \tag{5.31}$$

the BRST-transformation of (5.30) implies that

$$QV=\lambda'^{\mu j}\lambda'^{\nu k}\nabla_{\mu j}A'_{\nu k}+\overline{\lambda}'^{\dot\mu}_{j}\overline{\lambda}'^{\dot\nu}_{k}\overline{\nabla}^{k}_{\dot\mu}\overline{A}'^{k}_{\dot\nu}+\lambda'^{\mu j}\overline{\lambda}'^{\dot\nu}_{k}(\nabla_{\mu j}\overline{A}'^{k}_{\dot\nu}+\overline{\nabla}^{k}_{\dot\nu}A'_{\mu j}), \tag{5.32}$$

where $\nabla_{\mu j}$ and $\overline{\nabla}^{j}_{\dot\mu}$ are the covariant superspace derivatives in an AdS_4 background. So $QV=0$ implies that

$$\nabla_{\mu j}\overline{A}'^{k}_{\dot\nu}+\overline{\nabla}^{k}_{\dot\nu}A'_{\mu j}=\delta^k_j\sigma^m_{\mu\dot\nu}A_m,\quad e^{\phi}\nabla_{(\mu j}A'_{\nu k)}=\epsilon_{\mu\nu}A_{[jk]},\quad e^{-\phi}\overline{\nabla}^{(\dot\mu j}\overline{A}'^{\dot\nu k)}=\frac{1}{2}\epsilon^{\dot\mu\dot\nu}\epsilon^{hijk}A_{[hi]}, \tag{5.33}$$

for some superfields A_m and $A_{[jk]}$.

Although the equations of (5.33) are difficult to solve when written in terms of AdS_4 superspace variables, they can be simplified by performing a superconformal transformation

JHEP08(2007)011

from N=4 AdS_4 superspace into N=4 d=4 Minkowski superspace. A point $(y^m, \xi^{\mu j}, \overline{\xi}^{\dot{\mu}}_j)$ in AdS_4 superspace can be represented as

$$g_{AdS_4}(y, \xi, \overline{\xi}) = e^{y^m(P_m+K_m)+\xi^{\mu j}(Q_{\mu j}+S^k_\mu \delta_{jk})+\overline{\xi}^{\dot{\mu}}_j(\overline{Q}^j_{\dot{\mu}}+\overline{S}_{\dot{\mu}k}\delta^{jk})} \tag{5.34}$$

where $g(y, \xi, \overline{\xi})$ is an element of $PSU(2,2|4)$ whose bosonic generators for translations, conformal boosts, rotations, dilatations and SU(4) R-transformations are denoted respectively by $[P_m, K_m, M_{[mn]}, D, R^k_j]$, and whose fermionic generators for supersymmetry and superconformal transformations are denoted respectively by $[Q_{\mu j}, \overline{Q}^j_{\dot{\mu}}, S^j_\mu, \overline{S}_{\dot{\mu}j}]$. Under an N=4 superconformal transformation parameterized by the $PSU(2,2|4)$ element Ω,

$$g_{AdS_4}(y, \xi, \overline{\xi}) \rightarrow g'_{AdS_4}(y', \xi', \overline{\xi}') = \Omega \; g_{AdS_4}(y, \xi, \overline{\xi}) \; h(y, \xi, \overline{\xi}) \tag{5.35}$$

where

$$h = e^{c^m K_m + w^{mn} M_{[mn]} + a^j_k R^k_j + bD + \chi^\mu_j S^j_\mu + \overline{\chi}^{\dot{\mu} j}\overline{S}_{\dot{\mu}j}} \tag{5.36}$$

and the parameters $[c^m, w^{mn}, a^j_k, b, \chi^\mu_j, \overline{\chi}^{\dot{\mu}}_j]$ in (5.36) are chosen such that

$$g'_{AdS_4} = e^{y'^m(P_m+K_m)+\xi'^{\mu j}(Q_{\mu j}+S^k_\mu \delta_{jk})+\overline{\xi}'^{\dot{\mu}}_j(\overline{Q}^j_{\dot{\mu}}+\overline{S}_{\dot{\mu}k}\delta^{jk})} \tag{5.37}$$

for some $(y'^m(y, \xi, \overline{\xi}), \xi'^{\mu j}(y, \xi, \overline{\xi}), \overline{\xi}'^{\dot{\mu}}_j(y, \xi, \overline{\xi}))$.

Similarly, a point $(x^m, \theta^{\mu j}, \overline{\theta}^{\dot{\mu}}_j)$ in N=4 d=4 Minkowski superspace can be represented as

$$g_{\rm Mink}(x, \theta, \overline{\theta}) = e^{x^m P_m + \theta^{\mu j} Q_{\mu j} + \overline{\theta}^{\dot{\mu}}_j \overline{Q}^j_{\dot{\mu}}} \tag{5.38}$$

where under an N=4 superconformal transformation parameterized by Ω,

$$g_{\rm Mink}(x, \theta, \overline{\theta}) \rightarrow g'_{\rm Mink}(x', \theta', \overline{\theta}') = \Omega \; g_{\rm Mink}(x, \theta, \overline{\theta}) \; h(x, \theta, \overline{\theta}) \tag{5.39}$$

and the parameters $[c^m, w^{mn}, a^j_k, b, \chi^\mu_j, \overline{\chi}^{\dot{\mu}}_j]$ in h of (5.36) are now chosen such that $g'_{\rm Mink} = e^{x'^m P_m + \theta'^{\mu j} Q_{\mu j} + \overline{\theta}'^{\dot{\mu}}_j \overline{Q}^j_{\dot{\mu}}}$ for some $(x'^m(x, \theta, \overline{\theta}), \theta'^{\mu j}(x, \theta, \overline{\theta}), \overline{\theta}'^{\dot{\mu}}_j(x, \theta, \overline{\theta}))$.

To superconformally map N=4 AdS_4 superspace into N=4 d=4 Minkowski superspace, define

$$g_{\rm Mink}(x, \theta, \overline{\theta}) = g_{AdS_4}(y, \xi, \overline{\xi}) \; h(y, \xi, \overline{\xi}) \tag{5.40}$$

where the parameters $[c^m, w^{mn}, a^j_k, b, \chi^\mu_j, \overline{\chi}^{\dot{\mu}}_j]$ in h of (5.36) are chosen such that $g_{\rm Mink} = e^{x^m P_m + \theta^{\mu j} Q_{\mu j} + \overline{\theta}^{\dot{\mu}}_j \overline{Q}^j_{\dot{\mu}}}$ for some functions $(x^m(y, \xi, \overline{\xi}), \theta^{\mu j}(y, \xi, \overline{\xi}), \overline{\theta}^{\dot{\mu}}_j(y, \xi, \overline{\xi}))$. After writing the AdS_4 superspace variables $(y^m, \xi^{\mu j}, \overline{\xi}^{\dot{\mu}}_j)$ in terms of the Minkowski superspace variables $(x^m, \theta^{\mu j}, \overline{\theta}^{\dot{\mu}}_j)$ using this superconformal map, the superfield equations of (5.33) simplify to

$$D_{\mu j}\overline{A}'^k_{\dot{\nu}} + \overline{D}^k_{\dot{\nu}} A'_{\mu j} = \delta^k_j \sigma^m_{\mu\dot{\nu}} A_m, \quad e^\phi D_{(\mu j} A'_{\nu k)} = \epsilon_{\mu\nu} A_{[jk]}, \quad e^{-\phi}\overline{D}^{(\dot{\mu} j}\overline{A}'^{\dot{\nu} k)} = \frac{1}{2}\epsilon^{\dot{\mu}\dot{\nu}}\epsilon^{hijk} A_{[hi]}, \tag{5.41}$$

where $D_{\mu j}$ and $\overline{D}^j_{\dot{\mu}}$ are the flat superspace derivatives. So if one defines $A'_{\mu j}(x, \theta', \overline{\theta}') = e^{-\frac{\phi}{2}} A_{\mu j}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta} e^{\frac{\phi}{2}})$ and $\overline{A}'^j_{\dot{\mu}}(x, \theta', \overline{\theta}') = e^{\frac{\phi}{2}} \overline{A}^j_{\dot{\mu}}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta} e^{\frac{\phi}{2}})$ as in (5.25), one finds that

$$D_{\mu j}\overline{A}^k_{\dot{\nu}} + \overline{D}^k_{\dot{\nu}} A_{\mu j} = \delta^k_j \sigma^m_{\mu\dot{\nu}} A_m, \quad D_{(\mu j} A_{\nu k)} = \epsilon_{\mu\nu} A_{[jk]}, \quad \overline{D}^{(\dot{\mu} j}\overline{A}^{\dot{\nu} k)} = \frac{1}{2}\epsilon^{\dot{\mu}\dot{\nu}}\epsilon^{hijk} A_{[hi]}, \tag{5.42}$$

which are the same equations as (5.27). So the OSp(4|4)-invariant worldline action of (5.29) also describes N=4 d=4 super-Yang-Mills.

5.4 Equivalence with open topological A-model

It will now be shown that the worldline action of (5.29), which is based on the $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(4)}$ coset together with semi-pure spinors, is related by a field redefinition to the worldline action of (4.15), which is based on the $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,2)\times\mathrm{SO}(4)}$ coset together with unconstrained spinors. This field redefinition combines the four x's of the $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(4)}$ coset with the 12 components of the semi-pure spinors to form an unconstrained 16-component spinor which transforms covariantly like a twistor variable under $\mathrm{SO}(3,2)$ transformations. The construction of this AdS_4 twistor variable is very similar to the construction of the $AdS_5\times S^5$ twistor variable of subsection 3.2 in which the ten x's of the $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ coset were combined with the 22 components of the pure spinors to form two unconstrained 16-component spinors.

To construct the field redefinition, first decompose the $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(4)}$ coset as

$$g(x,\theta) = e^{\theta^{rj}T_{rj}}e^{x^m T_m} \equiv G(\theta)H(x) \tag{5.43}$$

where $G(\theta) = e^{\theta^{rj}T_{rj}}$ takes values in $\frac{\mathrm{OSp}(4|4)}{Sp(4)\times\mathrm{SO}(4)}$, $H(x) = e^{x^m T_m}$ takes values in $\frac{Sp(4)}{\mathrm{SO}(3,1)}$, and T_{rj} and T_m are the "supersymmetry" and "translation" generators of $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(4)}$.

Now define the twistor-like variable as

$$Z^{rj} = H^r_s \lambda'^{sj} \tag{5.44}$$

which combines the four x's in H^r_s with the 12 components of the semi-pure spinor λ'. Similarly, define the conjugate twistor-like variable as

$$Y_{jr} = (H^{-1})^s_r w'_{js}. \tag{5.45}$$

Using

$$J = \left(g^{-1}\frac{\partial}{\partial\tau}g\right) = \left(H^{-1}\frac{\partial}{\partial\tau}H\right) + H^{-1}\left(G^{-1}\frac{\partial}{\partial\tau}G\right)H, \tag{5.46}$$

one finds that

$$\begin{aligned} Y_{jr}\frac{\partial}{\partial\tau}Z^{rj} &= w'_{rj}\frac{\partial}{\partial\tau}\lambda'^{rj} + \left(H^{-1}\frac{\partial}{\partial\tau}H\right)^s_r (w'\lambda')^r_s \\ &= w'_{rj}\frac{\partial}{\partial\tau}\lambda'^{rj} + J^s_r(w'\lambda')^r_s - \left(G^{-1}\frac{\partial}{\partial\tau}G\right)^s_r (YZ)^r_s \\ &= w'_{rj}(\nabla\lambda')^{rj} + J^m(w'\sigma_m\lambda') - \left(G^{-1}\frac{\partial}{\partial\tau}G\right)^s_r (YZ)^r_s - \left(G^{-1}\frac{\partial}{\partial\tau}G\right)^k_j (YZ)^j_k, \end{aligned} \tag{5.47}$$

where $(w'\lambda')^r_s = w'_{js}\lambda'^{rj}$, $(w'\lambda')^j_k = (YZ)^j_k = Y_{kr}Z^{rj}$, $(w'\sigma^m\lambda') = (\sigma^m)^r_s w'_{rj}\lambda'^{sj}$, and $(\nabla\lambda')^{rj} = \frac{\partial}{\partial\tau}\lambda'^{rj} + \frac{1}{2}J^{mn}(\sigma_{mn}\lambda')^{rj} + J^j_k\lambda'^{rk}$. Furthermore,

$$\begin{aligned} (w'\sigma^{mn}\lambda')(w'\sigma_{mn}\lambda') &= w'\lambda')^s_r(w'\lambda')^r_s - (w'\sigma^m\lambda')(w'\sigma_m\lambda') \\ &= (YZ)^s_r(YZ)^r_s - (w'\sigma^m\lambda')(w'\sigma_m\lambda'). \end{aligned} \tag{5.48}$$

JHEP08(2007)011

Plugging (5.47) and (5.48) into the action of (5.29), and introducing an auxiliary variable P_m to write the $J_m J^m$ kinetic term in first-order form, one finds that the action of (5.29) can be written as

$$
\begin{aligned}
S = &\int d\tau [P_m J^m - P_m P^m + \epsilon_{rs} J^{rj} J^{sj} + Y_{jr}(\nabla Z)^{rj} \\
&+ (YZ)^k_j (YZ)^j_k - (YZ)^s_r (YZ)^r_s - J^m (w' \sigma_m \lambda') + (w' \sigma^m \lambda')(w' \sigma_m \lambda')] \qquad (5.49)\\
= &\int d\tau [P'_m (J^m - 2 w' \sigma^m \lambda') - P'_m P'^m + \epsilon_{rs} J^{rj} J^{sj} + Y_{jr} (\nabla Z)^{rj} \\
&+ (YZ)^k_j (YZ)^j_k - (YZ)^s_r (YZ)^r_s], \\
&\text{where } \ (\nabla Z)^{rj} = \frac{\partial}{\partial \tau} Z^{rj} + \left(G^{-1} \frac{\partial}{\partial \tau} G \right)^r_s Z^{sj} + \left(G^{-1} \frac{\partial}{\partial \tau} G \right)^j_k Z^{rk} \\
&\text{and } \ P'_m = P_m - (w' \sigma_m \lambda'). \qquad (5.50)
\end{aligned}
$$

Under the gauge transformation $\delta {w'}_{rj} = \xi^m (\sigma_m)^s_r {\lambda'}_{sj}$ of (5.21), (5.50) implies that

$$
\delta P'_m = \xi^n (\sigma_{mn})^s_r \lambda'^{rj} {\lambda'}_{sj}. \qquad (5.51)
$$

For generic values of λ'^{rj}, $\det(\delta P' / \delta \xi)$ is non-zero, so one can consistently gauge $P'_m = 0$. Moreover, it is expected that the Fadeev-Popov factor from this gauge-fixing of P'_m is cancelled by the measure factor which converts the four x's and 12 constrained λ''s into the 16 unconstrained Z^{rj}'s.

In the gauge $P'_m = 0$, the action of (5.49) reduces to

$$
S = \int d\tau [\epsilon_{rs} J^{rj} J^{sj} + Y_{rj} (\nabla Z)^{rj} + (YZ)^k_j (YZ)^j_k - (YZ)^s_r (YZ)^r_s], \qquad (5.52)
$$

where (5.46) implies that $\epsilon_{rs} J^{rj} J^{sj} = \epsilon_{rs} (G^{-1} \frac{\partial}{\partial \tau} G)^{rj} (G^{-1} \frac{\partial}{\partial \tau} G)^{sj}$. Since G parameterizes the coset $\frac{\mathrm{OSp}(4|4)}{\mathrm{SO}(3,2) \times \mathrm{SO}(4)}$, the worldline action of (5.52) is equivalent to the worldline action of (4.15) coming from the open topological A-model. And since the BRST cohomology of (5.29) describes d=4 N=4 super-Yang-Mills, this equivalence implies that the physical states in the open sector of the topological A-model are d=4 N=4 super-Yang-Mills states.

6. Conclusions

In this paper, a new limit of the $AdS_5 \times S^5$ sigma model was considered in which the vector components of the $PSU(2,2|4)$ metric $g_{ab} \to \infty$ and the superspace torsion $T_{\alpha\beta}{}^a \to 0$, while the spinor components of the $PSU(2,2|4)$ metric $g_{\alpha\widehat{\beta}}$ and the superspace torsion $T_{\alpha a}{}^{\widehat{\beta}}$ are held fixed. This is the opposite procedure from the flat space limit, and if $(T^b_{\alpha\beta} \eta_{ab})/(T^{\widehat{\beta}}_{\alpha a} \eta_{\beta\widehat{\beta}})$ is interpreted as the $AdS_5 \times S^5$ radius, it corresponds to taking this radius to zero.

In this limit, the $PSU(2,2|4)$ algebra deforms into an $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$ bosonic algebra with 32 abelian fermionic isometries, and the $AdS_5 \times S^5$ sigma model reduces to a linear topological A-model constructed from fermionic N=2 superfields. The bosonic components of these fermionic superfields involve twistor-like combinations of the x's and pure spinor ghosts, and the linear topological A-model can be interpreted as the limit of a $PSU(2,2|4)$-invariant non-linear topological A-model whose open string sector describes N=4 d=4 super-Yang-Mills.

These results have many parallels with the open-closed duality found by Gopakumar and Vafa which relates Chern-Simons theory and the resolved conifold [17]. In this open-closed duality, Chern-Simons theory is described by the open sector of a topological A-model [13], which is interpreted as a Coulomb branch of the closed string theory for the resolved conifold. As pointed out in [17] and [18], the Chern-Simons/conifold duality shares many features with the Yang-Mills/$AdS_5 \times S^5$ duality, suggesting that the Ooguri-Vafa worldsheet proof of Chern-Simons/conifold duality [18] might have a generalization to a worldsheet proof of the Maldacena conjecture.

However, before attempting a proof of Maldacena's conjecture using the results of this paper, one would need to understand better both the properties of the $T_{\alpha\beta}{}^a \to 0$ limit of the $AdS_5 \times S^5$ sigma model, and the properties of the open topological A-model for N=4 d=4 super-Yang-Mills.

For example, it is not clear that the $T_{\alpha\beta}{}^a \to 0$ limit of the sigma model can be interpreted as the small $AdS_5 \times S^5$ radius limit, and that a separate Coulomb branch is developed in this limit. Furthermore, although it was shown that the physical states of the open topological A-model describes N=4 d=4 super-Yang-Mills, it was not shown how to compute perturbative super-Yang-Mills scattering amplitudes using this A-model. Hopefully, the d=10 pure spinor formalism will provide some useful clues for computing these amplitudes. For example, if the d=10 pure spinor measure factor $\langle(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta)(\theta\gamma_{abc}\theta)\rangle = 1$ is dimensionally reduced to four dimensions, the field theory action for the open A-model

$$S = \langle VQV + \frac{2}{3}VVV\rangle \tag{6.1}$$

appears to correctly reproduce the N=4 d=4 super-Yang-Mills action [15, 16]. So using the interaction vertex from (6.1), it should be possible to at least compute 3-point super-Yang-Mills tree amplitudes with the open topological A-model. A much bigger challenge would be to compute 4-point tree amplitudes using the A-model, and perhaps the twistor-string methods of [14, 24, 25] will be useful in these computations.

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